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AN INTRODUCTION TO STABILITY THEORY OF DYNAMICAL SYSTEMS

Stephen Laurel Schey

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THESIS

AN INTRODUCTION TO
STABILITY THEORY OF DYNAMICAL SYSTEMS

by

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June 1973

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An Introduction to
Stability Theory of Dynamical Systems

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ABSTRACT

This thesis investigates stable, asymptotically stable and unstable dynamical systems from a topological point of view with direct application and interpretation to systems of differential equations. Knowledge of topology is not a prerequisite. Specifically, such concepts as Poisson and Liapunov stability as well as parallelizable and dispersive systems are investigated.

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INTRODUCTION

As it is currently available, stability theory of dynamical systems requires an extensive background in higher mathematics. The purpose of this thesis then is to make this theory accessible to the reader who does not have this background. It is assumed that the reader has taken courses in elementary differential equations, intermediate mathematical analysis, and linear algebra. All the topological properties needed are described in Section 1.

Sections 1 - 3 deal with developing basic concepts of dynamical systems. Sections 4 - 6 use these concepts in studying various kinds of stable or unstable systems. Topics of recursiveness and dispersiveness are also discussed.

Among the stable systems, Poisson and Liapunov stability will be discussed as well as non-wandering systems. Poisson instability, Lagrange instability, and complete instability will also be investigated. Topics in dispersiveness include parallelizable and dispersive systems. The relationship between several of these systems will be shown. One such relationship to be shown is that a parallelizable system is dispersive, completely unstable, Lagrange unstable and Poisson unstable.

Several important and revealing theorems will be presented. Among these are the theorems involving the Liapunov function which allows analysis of a system as to stability or instability without actually solving the system.

The book Stability Theory of Dynamical Systems by Bhatia and Szego will be used extensively throughout this thesis and its bibliography contains an excellent list of references.

I. TOPOLOGICAL PROPERTIES

The purpose of this section is to expose the reader to all of the topological properties needed to understand the material presented in this thesis. It is important that the reader develop a firm understanding and familiarity with these concepts to facilitate in understanding of the material to follow in later sections.

We begin with the concept of distance.

If X is a non-empty set, a metric or distance function, on X is a real-valued function d of ordered pairs of elements of X which satisfy three conditions. First, $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$. Second, $d(x,y) = d(y,x)$. This is called the symmetric property. Finally, $d(x,y) \leq d(x,z) + d(z,y)$. This property is called the triangle inequality. A metric space X is then a set X with a metric defined on it.

If x_0 is any point of a metric space and r is a positive real number, the open sphere $S(x_0,r)$ with center x_0 and radius r is the subset of X defined by $S(x_0,r) = \{x : d(x,x_0) < r\}$. Now a subset G of X is said to be open if and only if it is the union of open spheres. Since x_0 is clearly an element of $S(x_0,r)$, we say that $S(x_0,r)$ is a (spherical) neighborhood of x_0.

If A is a subset of a metric space X, a point x in X is called an accumulation point of A if each open sphere

centered on x contains at least one point of A different from x. The essential idea here is that the points of A different from x get arbitrarily close to x. Then a subset F of a metric space X is called a closed set if it contains all its accumulation points. Another characterization of a closed set is that a set $F \subset X$ is closed if and only if its complement in X is an open set. Then by definition, the whole space X and the empty set $\emptyset$ are open sets. Then since the empty set is the complement of the whole space, we have by our previous observation that the whole space X and the empty set $\emptyset$ are also closed sets.

Now, in any metric space, the union of any collection of open sets is itself an open set and the finite intersection of open sets is an open set. Taking set complements it follows that any intersection of closed sets is a closed set and the finite union of closed sets is a closed set.

If x_0 is a point in the metric space X and r is a non-negative real number, then the closed sphere $S[x_0,r]$ with center x_0 and radius r is the subset of X defined by $S[x_0,r] = \{x:d(x,x_0) \leq r\}$.

Let X be an arbitrary metric space and let A be a subset of X. A point in A is called an interior point of A if it is the center of some open sphere contained in A; and the interior of A, denoted by $\mathrm{int}_X A$, is the set of all its interior points. Symbolically, we have $\mathrm{int}_X A = \{x:x \in A \text{ and } S(x,r) \subset A \text{ for some } r\}$. Since the interior of a set is a union of open spheres, it follows

that $\mathrm{int}_X A$ is an open set. Hence we have the characterization that *a set* A *is open if and only if* $A = \mathrm{int}_X A$. We often write int A in place of $\mathrm{int}_X A$ whenever no confusion will result.

If X is a metric space and A is a subset of X, the *closure* of A in X, denoted by $\mathrm{cl}_X A$, is the union of A and the set of all its accumulation points. Intuitively, $\mathrm{cl}_X A$ is A itself together with all other points in X arbitrarily close to A. Since it is clear that $\mathrm{cl}_X A$ contains all its accumulation points, $\mathrm{cl}_X A$ is a closed set. Thus we have an additional characterization of a closed set: $A \subset X$ *is closed if and only if* $A = \mathrm{cl}_X A$. We often write cl A in place of $\mathrm{cl}_X A$.

Again let X be a metric space and let A be a subset of X. We will use the notation X-A to denote the complement of A in X. A point in X is called a *boundary point* of A if each open sphere centered on this point intersects both A and its complement X-A. Now the *boundary* of A, denoted bdy A, is the set of all the boundary points of A. It is easily shown that $\mathrm{bdy}\ A = \mathrm{cl}_X A \cap \mathrm{cl}_X(X-A)$. Moreover, the boundary of A is a closed set, and *a set* $A \subset X$ *is closed if and only if it contains all of its boundary points*.

Every metric space has, in addition to the properties already stated, two important "separation" properties. First the *Hausdorff* property: if x and y are two distinct points in X, then there exist disjoint open sets U_x and U_y

such that $x \epsilon U_x$ and $y \epsilon U_y$. Second, the *normal space* property: if A and B are disjoint closed sets of X, then there exist disjoint open sets U_A and U_B such that $A \subset U_A$ and $B \subset U_B$.

We now use these basic definitions to describe more advanced ideas in topology. We are interested in characterizing "compactness", "completeness" and "connectedness".

Let X be any metric space. A class $\{G_i\}$ of open subsets of X is said to be an *open cover* of X if each point in X belongs to at least one G_i, that is $\bigcup_i G_i = X$. A subclass of an open cover which is itself an open cover is called a *subcover*. Then a *compact space* is defined to be a metric space in which every open cover has a finite subcover. This becomes a familiar definition when one recalls the *Heine-Borel Theorem* which states that *if* $[a,b]$ *is a closed interval of real numbers and if* $\{G_x\}$, $x \epsilon [a,b]$ *is an open cover of* $[a,b]$, *then there exists a finite number of the* G_x*'s whose union contains* $[a,b]$. This is just the statement that closed intervals on the real line are compact sets.

A much used property of a compact space is that *any closed subspace of a compact space is itself compact*. Another useful property is that *any infinite sequence in a compact set contains a subsequence which converges to some point in the compact set*. This also becomes a familiar property when one recalls the *Bolzano-Weierstrass Theorem*: *Every bounded infinite set* A *of real numbers has at least one point of accumulation*. This theorem is just an application of

this property applied to the real line. These last two properties are very important and will be used extensively throughout this thesis.

To define the notion of a complete metric space we need the concept of convergence. If $\{x_n\}$ is a sequence in X, we say that $\{x_n\}$ is convergent to the point x in X if given any $\varepsilon > 0$, there exists a positive integer N such that $d(x_n,x) < \varepsilon$ whenever $n > N$. Then in particular we have $d(x_n,x_m) < \varepsilon$ whenever $n,m > N$. A sequence with this latter property is called a Cauchy sequence. Now, not every Cauchy sequence is a convergent sequence. For example, consider the following sequence in the metric space of rational numbers, {3., 3.1, 3.14, 3.141, 3.1415, ...}. Clearly, this is a Cauchy sequence but it is not convergent because it converges to π; but π is not a rational number. Motivated by this example we say that a metric space is complete if every Cauchy sequence in X is convergent. Completeness is related to compactness in that a set is compact if and only if it is complete and totally bounded. By totally bounded we mean that the space X is the union of a finite number of open spheres of radius less than r for each positive r.

Finally, we define the concept of connectedness. A connected space is a metric space X which cannot be represented as the union of two disjoint non-empty open sets. If $X = A \cup B$ where A and B are disjoint non-empty open sets, then $X-A = B$ and $X-B = A$ so A and B are also closed sets. Thus a space X is connected if and only if the only sets which are both open and closed are the whole space X and the

empty set $\emptyset$. This property is very important and it can be shown that the *intermediate value theorem* in calculus is equivalent to this property for intervals on the real line.

We now turn our attention to the concept of a *continuous function*. This concept has several different characterizations and we begin with the basic definition that looks like the definition used in elementary calculus.

Let f be a function from a metric space X with metric d to a metric space Y with metric d'. Then f is said to be *continuous at* $x_0 \varepsilon X$, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x \varepsilon X$, we have $d'(f(x),f(x_0)) < \varepsilon$ whenever $d(x,x_0) < \delta$. The function f is said to be *continuous* if it is continuous at each point of X. The definition we saw in calculus was for real-valued functions so the metrics just involved the usual absolute value for the real line.

Several other characterizations of a continuous function will be needed in this thesis and are stated here. *A function is continuous if and only if the inverse image of each open (closed) set is an open (closed) set*. Also *a function* f *is continuous if and only if for each sequence* $\{x_n\}$ *which converges to* x, *the sequence* $\{f(x_n)\}$ *converges to* f(x).

We now state some important properties of continuous functions.

> *A continuous real-valued function* f
> *defined on a compact set* A *attains its*
> *maximum and minimum values on* A *in the*

following sense: if $a = \inf\{f(x):x \varepsilon A\}$ *and* $b = \sup\{f(x):x \varepsilon A\}$, *then there exists* $x_1,x_2 \varepsilon A$ *such that* $f(x_1) = a$, *and* $f(x_2) = b$.

The continuous image of a compact (*connected*) *set is compact* (*connected*).

Next, we define the concept of a homeomorphism. A *homeomorphism* is a one-to-one continuous function from one metric space onto another metric space such that the inverse of this function is also continuous. (A *one-to-one function* is a function which assigns different images to different elements in its domain. A function f from a metric space X to a metric space Y is *onto* if the image of X under f is the entire space Y. If a function is one-to-one and onto, then its inverse is well defined.)

We have already stated the definition of a compact space. A closely related idea is that of a locally compact space. As its name suggests, a metric space X is *locally compact* if and only if each point of X has at least one compact neighborhood. Hence, a compact space is automatically locally compact.

One of the most important properties of a locally compact space is that through the addition of one point, usually referred to as the ideal point, the space can be extended to a compact space. This extension is known as the one-point

compactification. The one-point compactification of a metric space X is the space $X' = X \cup \{\omega\}$ where the open sets in X' are all the open sets in X together with all subsets U of X' such that X'-U is a compact subset of X. In other words, a set U is open in X' if and only if $U \cap X$ is open and whenever $\omega \in U$, X-U is compact. As this is an important fact, we give a formal proof. We must show that X' is compact and that unions and finite intersections of open sets are open.

THEOREM: The one-point compactification X' of a metric space X is compact.

PROOF: Finite intersections and arbitrary unions of open sets in X' intersect X in open sets. If ω is a member of the intersection of two open subsets of X', then the complement of the intersection is the union of two closed compact subsets of X and is therefore closed and compact. Hence the intersection of two open subsets is open in X'. If ω belongs to the union of the members of a family of open subsets of X', then ω belongs to some member U of the family. Moreover the complement of the union is a closed subset of the compact set X-U and is therefore closed and compact. Hence the union of open sets is open in X'. Consequently, arbitrary unions and finite intersections of open sets in X' are open. Next, let $\mathcal{U}$ be an open covering of X'. Then ω is a member of some U in $\mathcal{U}$ and X-U is compact. Hence

there is a finite subcover of $\mathcal{U}$ which covers $X-U$ and thus this finite subcover together with U is a finite subcover of X'. Thus X' is compact. //

Our final topic in this section of topological prerequisites is the concept of product spaces. A finite product space is the cartesian product of two metric spaces. In this thesis, one of the metric spaces will usually be the real line R; consequently if X is a metric space, we have the product space to be $X \times R$. Hence any point in the product space will be written (x,t) where $x \varepsilon X$ and $t \varepsilon R$. Now the open sets in the product space are just the cartesian products of open sets in the metric spaces. That is if U is an open set in X and V is an open set in R, then $U \times V$ is an open set in $X \times R$, and every open set in $X \times R$ is of that form. If d is the metric for X, we define the metric d' on $X \times R$ in the following manner. If $x,y \varepsilon X \times R$, then there are elements $a,b \varepsilon X$ and $r,s \varepsilon R$ such that $x = (a,r)$ and $y = (b,s)$. Then $d'(x,y) = d(a,b) + |r - s|$. The spaces X and R are called the coordinate spaces of the product and the functions P_X and P_R, which carry the point (x,t) of $X \times R$ into x and t respectively, are called the projections into the coordinate spaces. It is easily shown that the projection mappings are continuous functions and that their inverses are also continuous functions. Moreover $f:Y \to X$ is continuous if and only if each of the compositions $P_X \circ f$ and $P_Y \circ f$ is continuous, for any metric space Y.

A function $f = f(x,t)$ defined on the product space is continuous in x if and only if for each t the function $f(\ ,t)$ whose value at x is $f(x,t)$, is continuous. Similarly, $f(x,t)$ is continuous in t if and only if for each $x \epsilon X$, the function $f(x, \ )$ such that $f(x, \ )(t) = f(x,t)$, is continuous. Finally if f is continuous on the product space then f is continuous in both x and t by our previous remarks. However, the converse is not, in general, true.

II. DYNAMICAL SYSTEMS

We seek to motivate the concept of dynamical systems by some examples of differential equations. But first we need the existence and uniqueness theorem associated with such systems.

We consider systems of first-order differential equations written in the form

$$\frac{dx_1}{dt} = f_1(t;x_1,\dots,x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(t;x_1,\dots,x_n).$$

Such systems frequently arise out of physical problems where the independent variable t represents time. Then a solution of the above system describes the "trajectory" of a particle moving in n-space. To see this more clearly, we look at the following two examples.

Consider the following system defined in R^3:

$$\frac{dx_1}{dt} = -\sin t$$

$$\frac{dx_2}{dt} = \cos t$$

$$\frac{dx_3}{dt} = 1.$$

Then the general solution to this system is given by

$$x_1 = \cos t + c_1$$

$$x_2 = \sin t + c_2$$

$$x_3 = t + c_3$$

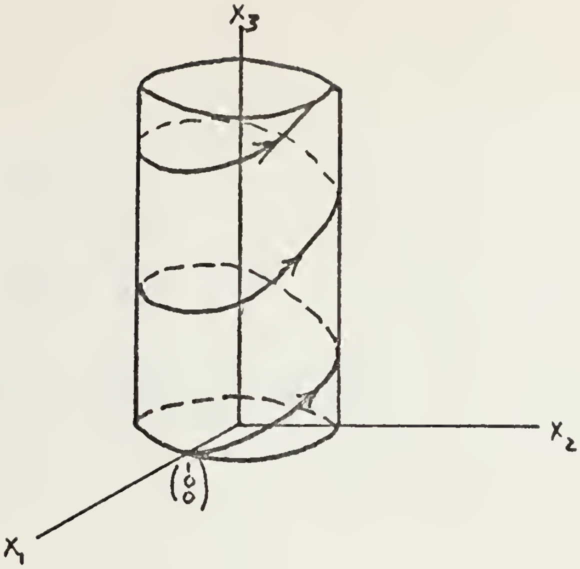
where the c's are all constants of integration. Suppose we specify that this solution passes through the point $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$ at time $t = 0$. Then the solution is given by

$$x_1 = \cos t$$

$$x_2 = \sin t$$

$$x_3 = t.$$

Such a solution describes the "trajectory" sketched below.



It is clear that at any time, we may find the exact position of the particle. The same is true of the following example.

Consider the system defined in R^3 given by

$$\frac{dx_1}{dt} = 2$$

$$\frac{dx_2}{dt} = 2t$$

$$\frac{dx_3}{dt} = t^2$$

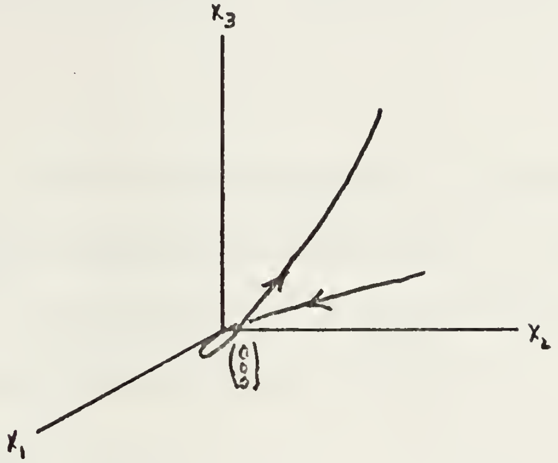
where we specify that its solution passes through the point $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ at time $t = 0$, Then this system has the solution given by

$$x_1 = 2t$$

$$x_2 = t^2$$

$$x_3 = \frac{t^3}{3}$$

Its "trajectory" is sketched below.



Thus an initial value problem for such a system consists of selecting an initial time $t = t_0$ and a point X_0 in R^n, and then seeking a solution whose graph passes through X_0 at time $t = t_0$.

By setting $X(t) = (x_1(t),\ldots,x_n(t))$,

$$f(t,X) = (f_1(t,X),\ldots,f_n(t,X)),$$

this system may be rewritten as

$$\frac{dX}{dt} = f(t,X), \quad \text{where} \quad \frac{dX}{dt} = (x_1'(t),\ldots,x_n'(t)).$$

We next want to examine conditions under which unique solutions exist for an initial value problem of the system.

Suppose that $f = f(t,X)$ is bounded and continuous in a region D of R^{n+1}. Furthermore, assume that f satisfies a *Lipschitz condition* in D; that is there exists a non-negative real number b such that whenever (t,X_1) and (t,X_2) are any two points in R, $d(f(t,X_1), f(t,X_2)) \leq bd(X_1,X_2)$ where d is the ordinary Euclidean metric. Then we have the following theorem.

> *Let* $f = f(t,X)$ *be continuous, bounded, and satisfy a Lipschitz condition in a region* D *of* R^{n+1}, *and let* (t_0,X_0) *be any point in* D. *Then the initial value problem*
>
> $$\frac{dX}{dt} = f(t,X)$$
>
> $$X(t_0) = X_0$$
>
> *has a unique solution on an interval* I *of* R *and the graph of the solution lies in* D.

We will be interested mainly in systems defined on all of R. Then by ensuring that f is continuous and bounded on all of R^{n+1} and satisfies a global Lipschitz condition, we will have existence and uniqueness of solutions to the above system.

One elegant and simple proof of the above theorem requires a knowledge of contraction mappings and some theory of fixed points and will not be presented here [Kreider, Kuller, Ostberg, p. 385].

A classical proof may be found in *Theory of Ordinary Differential Equations* by Coddington and Levinson.

Of critical importance is the question: what happens to the solution if small changes are made in the function f or in the initial conditions t_0 and X_0? In most physical problems whose solutions are obtained from such a system, experimental error and estimations of physical constants may affect the initial conditions, and the presence of variable parameters, such as temperature or density, may affect f. We would hope that such small variations would not cause a drastic variation in the solution. It can be shown that these unique solutions guaranteed by the theorem depend continuously on f and on t_0 and X_0 for a certain amount of time; that is small changes in f or in t_0 and X_0 produce small changes in the solution. More precisely,

> *Let* $f = f(t,X)$ *be continuous, bounded, and satisfy a Lipschitz condition in a domain* D *of the* $(n + 1)$-*dimensional* (t,X) *space, and suppose* X *is a solution of the system on an interval* I: $a \leq t \leq b$. *There exists a* $\delta > 0$ *such that for any* (t_1,X_0) *where* $a < t_1 < b$, *and* $d(X_0,X(t_1)) < \delta$ *there exists a unique solution* Y *on* I *with* $Y(t_2,t_1,X_0) = X_0$ *where* $a < t_2 < b$. *Moreover,* Y *is continuous in* D.

(Proofs of the continuity of solutions may be found in Kreider, Kuller, Ostberg, p. 390 and Coddington, Levinson, p. 22.)

What happens to these solutions over an extended period of time involves the theory of stability which is the primary focus of this thesis.

We will be interested mainly in first-order autonomous systems, that is, "time independent" systems. Such a system may be written in the form

$$\dot{X} = f(X) \quad \text{where } \dot{X} \text{ denotes } \frac{dX}{dt}$$

$$X(t_0) = X_0.$$

The following examples provide some insight into such autonomous systems.

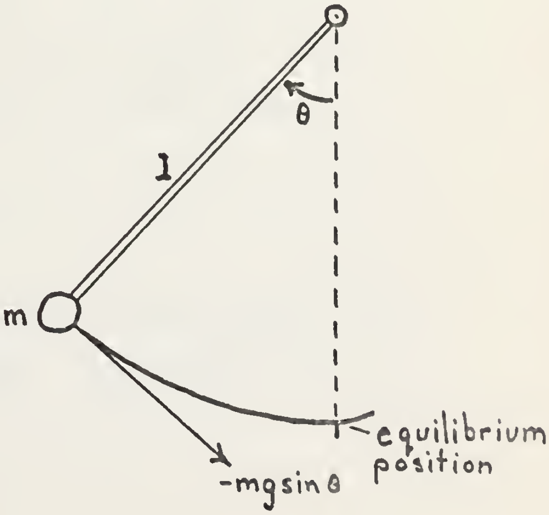
EXAMPLE 2.1 Consider a simple pendulum of mass m swinging without friction on a weightless rod of length l. By Newton's second law, the equation of motion of the pendulum is given by

$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2}$$

or

$$\ddot{\theta} = -g/l \sin\theta$$

By setting $g/l = k^2$ and $\dot{\theta} = \omega$ we obtain the following system:



$$\dot{\theta} = \omega$$

$$\dot{\omega} = -k^2 \sin \theta.$$

It can be shown [Kreider , Kuller, Ostberg, p. 397] that the general solution to this system is

$$\omega^2 = 2k^2 \cos \theta + c$$

where $c \geq -2k^2$ is a constant depending on the initial conditions.

Graphing several of these solutions in the $\theta\omega$-plane we have the following diagram.

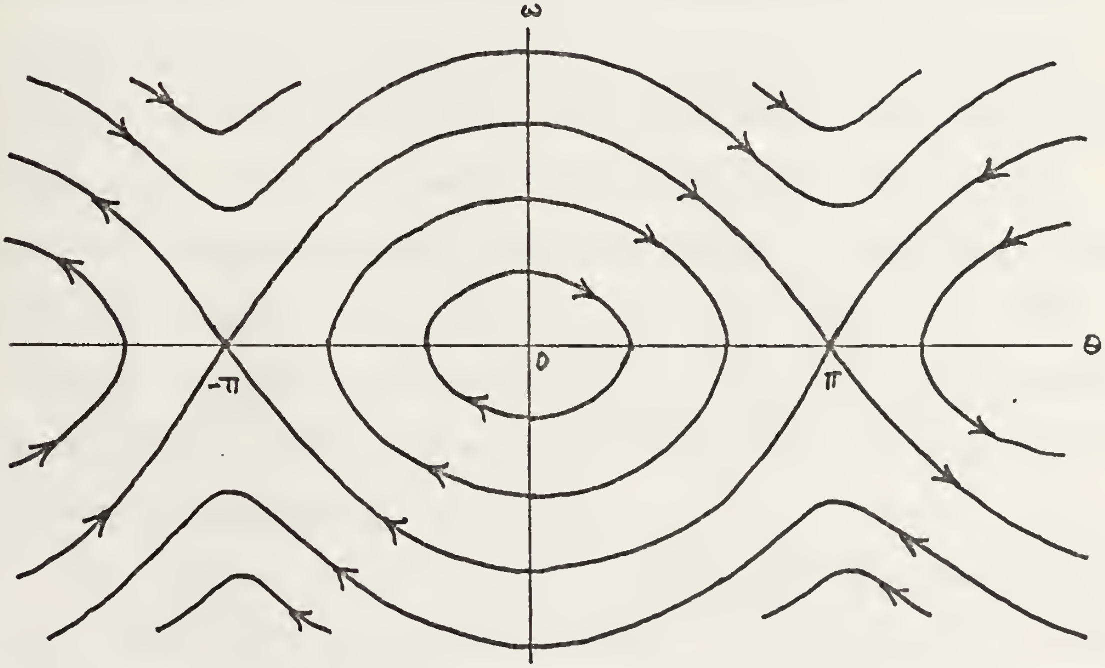


FIGURE 2.1

To see how this diagram was obtained, we see that ω is the angular velocity. Suppose $c = -2k^2$. Then when $\theta = 0$,

$\omega = 0$ and hence $\dot{\theta}$ and $\dot{\omega}$ are both zero. Thus the system is not changing with time. The point $\binom{0}{0}$ is the complete trajectory for this case. Physically this shows that if the mass is started at the equilibrium position with no angular velocity, the mass will remain at the equilibrium position indefinitely. Suppose $c = 2k^2$ and the system is started at $\theta = \pi$. Then again $\omega = 0$ and hence $\dot{\theta}$ and $\dot{\omega}$ are both unchanging with time and the point $\binom{\pi}{0}$ describes the complete trajectory. Physically this means that if the mass is balanced vertically above the equilibrium point with no angular velocity, it will also remain in that position indefinitely. Clearly the same analysis can be made for $\theta = \pm n\pi$ which is exactly what we would expect. If this system is started at any other angle θ, it will have an initial velocity which will cause the mass to approach the angle $\theta = \pi$ or $\theta = -\pi$ and it will get arbitrarily close but will never actually reach that point. Neither will this mass swing back away from this point but rather it will continue to approach the angle $\theta = \pi$ or $\theta = -\pi$ as time goes to infinity. This is the physical meaning of the trajectories between the points $\binom{-\pi}{0}$ and $\binom{\pi}{0}$.

Now suppose $-2k^2 < c < 2k^2$. From the previous remarks it is clear that the mass will have a velocity at $\theta = 0$ but its velocity is not great enough to permit it to approach the angle $\theta = \pi$ or $\theta = -\pi$. We would expect in this case that the mass would swing back and forth across the equilibrium point always reaching the same angle on either side

of the equilibrium point. This is the physical interpretation of the elliptical trajectories shown in the diagram.

Finally, suppose $c > 2k^2$. Then ω^2 is always positive which means that ω is always positive or always negative. If ω is always positive, then $\dot{\theta}$ is always positive and hence θ is always increasing. A similar statement follows if ω is always negative. Physically this is the case when the mass has been given enough initial velocity to swing completely over the top position. In a frictionless environment we would expect this behavior to continue indefinitely. Such is the physical interpretation of the top and bottom trajectories in the diagram.

To see the direction of a particle along any trajectory, pick an angle θ. For $0 < \theta < \pi$, we see that $\dot{\omega} < 0$ so ω is decreasing. For $-\pi < \theta < 0$, we see that $\dot{\omega} > 0$ so ω is increasing. This completes our analysis of the diagram.

It is noted that in this example the solution is not given as a function of time. To obtain such would mean the difficult task of solving a non-linear differential equation. This is not any more informative since the movement of a point along any trajectory with increasing time can easily be determined from the system describing the motion as we have just observed.

A more elementary approach to the problem is by observing that for small angles θ, $\sin \theta$ is approximated by θ. Thus our system would be given by

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -k^2\theta.$$

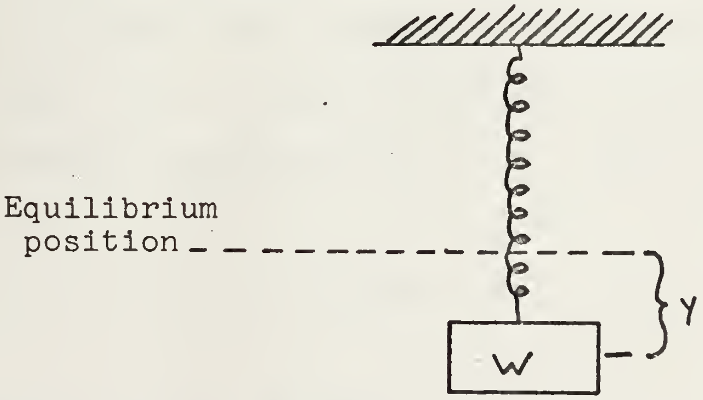
It can be shown that the solution to this system is

$$\theta = A\sin kt + B\cos kt$$

$$\omega = Ak\cos kt - Bk\sin kt$$

where A and B are constants depending on the initial conditions [Spiegel, p. 210]. It is seen that this solution describes simple harmonic motion and closely approximates our trajectories for small c where $-2k^2 < c < 2k^2$.

EXAMPLE 2.2 We consider the following vibrating spring problem.



The equation of motion is given by:

$$\frac{W\ddot{y}}{g} + b\dot{y} + ky = 0$$

where W is the weight of the mass, g is the force of gravity, b is the damping constant and k is the proportionality constant of the spring. Suppose W = 6 lbs, g = 32 lbs, k = 12 and $b \geq 0$ is left as an undetermined constant for now.

Then it can be shown [Spiegel, p. 195] that the general solution for y is

$$y = e^{-\frac{8}{3}bt}\left(A\cos\left(\frac{8}{3}\sqrt{9 - b^2}\ t\right) + B\sin\left(\frac{8}{3}\sqrt{9 - b^2}\ t\right)\right).$$

where the constants A and B depend on the initial conditions.

If we specify the initial conditions to be $y(0) = 1/3$ and $\dot{y}(0) = 0$, we have the solution,

$$y = \left(\frac{1}{9}\right)e^{-\frac{8}{3}bt}\left(3\cos\left(\frac{8}{3}\sqrt{9 - b^2}\ t\right) + \frac{b}{\sqrt{9-b^2}}\sin\left(\frac{8}{3}\sqrt{9-b^2}\ t\right)\right)$$

To make this example more explicit we will consider two cases for b. First an undamped system with b = 0 and then a damped system with b = 1.5. With b = 0, we have

$$y = \frac{1}{3}\cos\ 8t$$

With b = 1.5, we have

$$y = \frac{1}{9}e^{-4t}(3\cos 4\sqrt{3}\ t + \sqrt{3}\ \sin 4\sqrt{3}\ t)$$

By use of a trigonometric identity, this last equation may

be written

$$y = \frac{2\sqrt{3}}{9} e^{-4t} \sin\left(4\sqrt{3}\, t + \frac{\pi}{3}\right)$$

Now that we have the solutions, we change this second-order problem into a system of first-order equations. Let $X = \binom{x}{y}$ and $\dot{y} = x$. Then for $b = 0$

$$\dot{X} = f(X) = \begin{pmatrix} \frac{-kg}{W}\, y \\ x \end{pmatrix}$$

$$X(0) = \binom{0}{1/3}$$

For notation, let the initial point $\binom{0}{1/3} = X_0$. Then the solution to this system is

$$X(X_0, t) = \begin{pmatrix} -\frac{8}{3}\sin 8t \\ \frac{1}{3}\cos 8t \end{pmatrix}.$$

For $b = 1.5$, we have

$$\dot{X} = f(X) = \begin{pmatrix} \frac{-bgx}{W} - \frac{-kgy}{W} \\ x \end{pmatrix}$$

$$X(0) = \binom{0}{1/3}$$

and the solution to this system is

$$X(X_0,t) = \begin{pmatrix} \frac{8}{3}e^{-4t}\cos(4\sqrt{3}t+\frac{\pi}{3}) - \frac{8}{3}\sqrt{3}e^{-4t}\sin(4\sqrt{3}t+\frac{\pi}{3}) \\ \\ \frac{2}{9}\sqrt{3}e^{-4t}\sin(4\sqrt{3}t + \frac{\pi}{3}) \end{pmatrix}$$

It is clear that as $t \to \infty$, $X(X_0,t) \to \binom{0}{0}$ which we would intuitively expect from a damped spring.

We sketch these solutions on the x,y-plane.

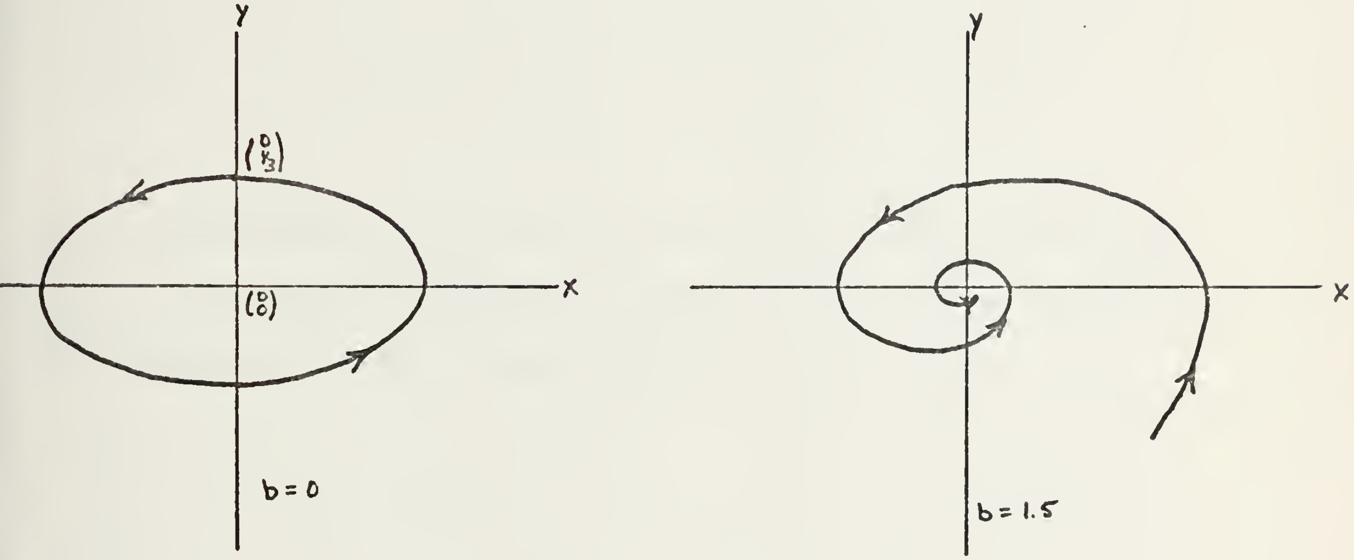


FIGURE 2.2

For the undamped problem we see that its trajectory describes simple harmonic motion. It continues to oscillate indefinitely with the same maximum displacement from the equilibrium position. On the other hand, the displacement and its rate of change both decrease with increasing time for the damped problem.

Since we have two different systems, it is necessary to

plot their solutions on two different planes. This helps to avoid any misunderstanding in our future work.

EXAMPLE 2.3 Consider the second-order differential equation $\ddot{y} + y = 0$, where y is considered as a function of time. By setting $x = \dot{y}$, this equation may be converted to the system

$$\dot{x} = -y$$

$$\dot{y} = x$$

Now considering $X = \binom{x}{y}$ with the initial condition that at $t = 0$, the solution passes through the point $\binom{1}{0}$, we have the following initial value problem

$$\dot{X} = f(X)$$

$$X(0) = \binom{1}{0}.$$

where f is the linear function defined by $f(t,x,y) = \binom{f_1(t,x,y)}{f_2(t,x,y)} = \binom{-y}{x}$. A simple check shows that this system has the solution

$$X(t) = \binom{\cos t}{\sin t}.$$

If we let X_0 denote the point $\binom{1}{0}$, then X_0 denotes the

initial condition. Further since the solution depends on this initial condition, the solution X(t) is more properly stated $X(X_0,t) = \binom{\cos t}{\sin t}$.

There are three observations we wish to make about this solution. First, we see that $X(X_0,0) = X_0$. Next, $X(X(X_0,t_2),t_1) = X(X_0,t_1 + t_2)$.

This is true since $X(X(X_0,t_2),t_1) = X(\binom{\cos t_2}{\sin t_2},t_1)$.

The term on the right is the solution of the system which has the initial condition $Y_0 = \binom{\cos t_2}{\sin t_2}$. Its solution is

$X(Y_0,t) = \binom{\cos t_2 \cos t - \sin t_2 \sin t}{\cos t_2 \sin t + \sin t_2 \cos t}$. Hence

$X(Y_0,t_1) = \binom{\cos t_2 \cos t_1 - \sin t_2 \sin t_1}{\cos t_2 \sin t_1 + \sin t_2 \cos t_1}$. Using a trigonometric identity we have

$X(Y_0,t_1) = \binom{\cos (t_1 + t_2)}{\sin (t_1 + t_2)}$. Thus

$X(X(X_0,t_2),t_1) = X(Y_0,t_1) = X(X_0,t_1 + t_2)$. Finally $X(X_0,t)$ is continuous because both of its components are continuous.

It is clear that the graph of the solution is a circle in the x,y-plane, and if we consider the graph to be the trajectory of the point X_0, that point moves counter-clockwise with increasing time. This can be seen by choosing some positive value for x. Then $\dot{y}$ is positive so y is increasing.

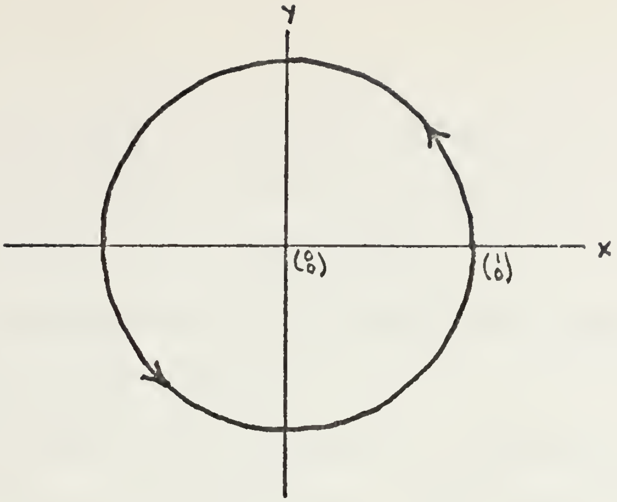


FIGURE 2.3

If we alter the initial condition to $X_0 = \binom{a}{b}$, the solution is $X(X_0,t) = \begin{pmatrix} a\cos t - b\sin t \\ a\sin t + b\cos t \end{pmatrix}$ and its trajectory is a circle passing through the point X_0.

Now referring to Example 2.3 we see that the solution $X(X_0,t)$ is a map from the product space $R^2 \times R$ into the space R^2. We note also that R^2 is a metric space. The three observations made about the solution are precisely the properties defining a dynamical system. The fact that the system in Example 2.3 does define a dynamical system will be made clear by the following definition where in this case, the map π is defined by $\pi(x,t) = X(x,t)$ where x denotes a point in R^2. In all further work, X denotes an arbitrary metric space.

DEFINITION 2.4 A _dynamical_ _system_ on X is the triplet (X,R,π), where π is a map from the product space $X \times R$ into the space X satisfying the following axioms:

1. For every x in X, $\pi(x,0) = x$,

2. For every x in X and t_1, t_2 in R,

$\pi(\pi(x,t_1),t_2) = \pi(x,t_1 + t_2)$, and

3. The map π is continuous.

The space X and the map π are called the phase space and the phase map, respectively.

From this point on, whenever no confusion will result, the symbol π will be suppressed. Thus the image under π of the point (x,t) will be written more simply as xt. The first two axioms then become

1. For every x in X, $x0 = x$,
2. For every x in X and t_1, t_2 in R,

$xt_1(t_2) = x(t_1 + t_2)$.

Similarly, if $M \subset X$ and $A \subset R$, MA is the set $\{xt: x \in M \text{ and } t \in A\}$.

The phase map determines two other maps when either x or t is fixed. For fixed t in R the map $\pi^t: X \to X$ defined by $\pi^t(x) = xt$ is called a transition; for fixed x in X the map $\pi_x: R \to X$ defined by $\pi_x(t) = xt$ is called a motion through x.

THEOREM 2.5 For each t in R, π^t is a homeomorphism on X onto itself.

PROOF For any t in R, π^t is clearly continuous because π is continuous. π^t is injective since $xt = yt$ implies $x = y$ because $x = x0 = x(t - t) = xt(-t) = yt(-t) = y(t - t) = y0 = y$. To see that π^t is surjective, let $y \in X$ be arbitrary.

Then $\pi^t(x) = y$ where $x = y(-t)$. Finally to show that π^t has a continuous inverse, it is sufficient to show that π^{-t} is the inverse of π^t. To see this we note that if π^t and π^s are two transitions, the composition $\pi^t \circ \pi^s$ is the transition π^{t+s}: for any x in X,

$$\pi^t \circ \pi^s(x) = \pi^t(\pi^s(x)) = \pi^t(xs) = xs(t) = x(s+t) = \pi^{s+t}(x).$$

It is also clear that π^0 is the identity transition because for any x in X, $\pi^0(x) = x0 = x$. Now since $\pi^{-t} \circ \pi^t = \pi^{t-t} = \pi^0$, the transition π^{-t} is the inverse of π^t.

REMARK 2.6 For any x in X and $[a,b] \subset R$, the set $x[a,b]$ is compact and connected.

PROOF The set $\{x\}$ is compact and connected in X and $[a,b]$ is compact and connected in R. Thus $\{x\}\times[a,b]$ is compact and connected. Then since $x[a,b]$ is the continuous image of this set, $x[a,b]$ is compact and connected. //

REMARK 2.7 Observe that the transitions π^t, t in R, form a commutative group with the group operation being the composition of transitions.

PROOF The closure property is trivial for if t,s are in R, $t + s$ is in R and $\pi^t \circ \pi^s = \pi^{t+s}$ as we have already observed.

Associative: $\pi^t \circ (\pi^s \circ \pi^v) = \pi^t \circ \pi^{s+v} = \pi^{t+s+v}$

$= \pi^{t+s} \circ \pi^v = (\pi^t \circ \pi^s) \circ \pi^v$

Commutative: $\pi^t \circ \pi^s(x) = \pi^t(\pi^s(x)) = \pi^t(xs) = (xs)t$

$= x(s+t) = x(t+s) = (xt)s = \pi^s(xt)$

$= \pi^s(\pi^t(x)) = \pi^s \circ \pi^t(x)$

Identity: The identity is π^0 since $\pi^t \circ \pi^0 = \pi^{t+0}$

$= \pi^t = \pi^{0+t} = \pi^0 \circ \pi^t$

Inverses: The inverse of π^t is π^{-t} since

$\pi^t \circ \pi^{-t} = \pi^0$.

We now give a precise definition of what we have been calling trajectories.

2.7 DEFINITION The maps $\gamma(x)$, $\gamma^+(x)$, and $\gamma^-(x)$ from X into 2^X are defined, for any x in X,

$$\gamma(x) = \{xt: t \in R\},$$

$$\gamma^+(x) = \{xt: t \in R^+\},$$

$$\gamma^-(x) = \{xt: t \in R^-\}.$$

For any x in X, the sets $\gamma(x)$, $\gamma^{+}(x)$, and $\gamma^{-}(x)$ are called respectively the trajectory, the positive semi-trajectory and the negative semi-trajectory. Note that for any x in X, $\gamma(x) = xR$.

If two trajectories γ_1 and γ_2 have a point in common, then the trajectories are identical. This fact is guaranteed by the existence and uniqueness theorem. For if $x \varepsilon \gamma_1$ and $x \varepsilon \gamma_2$, this theorem states that there exists a unique solution passing through the point x. Hence $\gamma_1 = \gamma_2$. This means that two different trajectories can never cross each other. This is an important fact and is very useful in diagramming dynamical systems.

Another example of a dynamical system which will prove to be useful is a dynamical system defined on a torus. Consider a differential system in the plane

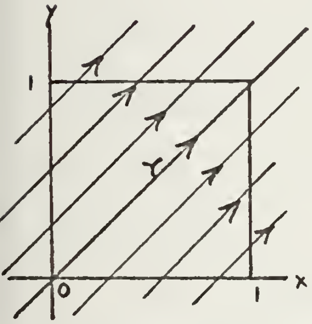
$$\dot{X} = f(X)$$

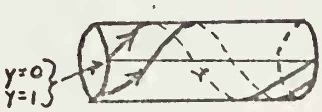
where f satisfies the hypotheses of the existence and uniqueness theorem. Set $X = \binom{x}{y}$ and $f = \binom{f_1}{f_2}$ and assume that the functions f_1 and f_2 are periodic with period 1 in each of the variables x and y. Then

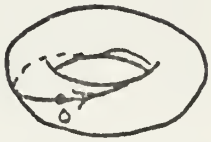
$$f_i(x,y) = f_i(x+1,y) = f_i(x,y+1), \quad i = 1,2$$

We take the square $\{(x,y): 0 \leq x < 1, 0 \leq y < 1\}$ and identify the sides x = 0 and x = 1 and the sides y = 0 and

y = 1. Then the points (0,0), (0,1), (1,0) and (1,1) are identified. Then we project the plane onto the torus by identifying any point (x,y) in the plane with a point $(\bar{x},\bar{y})$ of the torus where $\bar{x} = x \pmod 1$ and $\bar{y} = y \pmod 1$. Then the trajectories in the plane projected onto the torus yield trajectories of a dynamical system on the torus.







It is clear from Example 2.3 that if we pick any point x on the trajectory, xt is also on the trajectory for any t in R. In Example 2.2 if we pick a point x on any trajectory and consider the positive semi-trajectory, then xt belongs to $\gamma^+(x)$ for any $t \in R^+$. The same results can be seen for the negative semi-trajectory. This observation leads us naturally into the following concept of invariance.

2.8 DEFINITION A set $M \subset X$ is called invariant whenever

$$xt \in M \text{ for all } x \in M \text{ and } t \in R.$$

It is called positively invariant whenever this holds with

R replaced by R^+ and is called *negatively invariant* if it holds with R replaced by R^-.

From this definition we see that a set $M \subset X$ is invariant if and only if it is both positively and negatively invariant for if M is invariant, we let x be an arbitrary element of M. Then $xt \in M$ for all $t \in R$ and in particular for $t \in R^+$, $xt \in M$ and for $t \in R^-$, $xt \in M$. Thus M is both positively and negatively invariant. If M is both positively and negatively invariant, then for any $t \in R^+$, $xt \in M$ and for any $t \in R^-$, $xt \in M$. Thus $xt \in M$ for all $t \in R$ and M is invariant.

This concept leads us to some interesting and important properties associated with such sets.

THEOREM 2.9 *Let* $\{M_i\}$ *be a collection of positively invariant, negatively invariant or invariant subsets of* X. *Then their intersection and their union have the same property. A set* $M \subset X$ *is positively invariant if and only if the set* $X-M$ *is negatively invariant.* M *is invariant if and only if* $X-M$ *is invariant.*

PROOF Let the sets M_i be positively invariant. Let $A = \cup M_i$ and $B = \cap M_i$. For any $x \in A$, $x \in M_i$ for some i. Thus $xt \in M_i$ for all $t \in R^+$ since M_i is positively invariant. Then $xt \in A$ for all $t \in R^+$. Hence A is positively invariant. Let $x \in B$. Then $x \in M_i$ for all i. Then since each M_i is positively invariant, $xt \in M_i$ for all $t \in R^+$ and all i.

Thus $xt \in B$ for all $t \in R^+$ and B is positively invariant. The proofs for the negatively invariant and invariant cases are entirely analogous. Let M be positively invariant. Let $x \in X-M$ and $t \in R^-$ and suppose $xt \in M$. Since $-t \in R^+$, $xt(-t) \in M$. But $xt(-t) = x0 = x$ which implies $x \in M$ and we have a contradiction. Thus $xt \in X-M$ and X-M is negatively invariant. The proofs of the converse and the third part are entirely similar. //

THEOREM 2.10 *Let* $M \subset X$ *be positively invariant, negatively invariant or invariant. Then the following statements are true.*

1. *The closure of* M, cl M, *and its interior*, int M, *have the same property*,

2. *Each of its components has the same property*.

If M *is invariant, then so is its boundary* bdy M.

PROOF 1. Consider the case of invariance. Let $x \in \text{cl } M$ and $t \in R$. Then there is a sequence $\{x_n\}$ in M such that $x_n \to x$. By the invariance of M, $x_n t \in M$ for all $t \in R$ and each n. Then by continuity we have that $x_n t \to xt$. Thus $xt \in \text{cl } M$ and cl M is positively invariant. The proofs for the negatively invariant and positively invariant cases are entirely analogous.

Assume M is positively invariant. Then X-M and hence cl (X-M) are negatively invariant. Consequently int M = X - cl (X-M) is positively invariant. The proof for M negatively invariant is entirely similar.

2. Let M be invariant. Now $M = \bigcup M_i$ where the M_i's are disjoint components of M. Suppose M_i is not invariant for some i. Then there exists $x \in M_i$ and $t \in R$ such that $xt \notin M_i$. Now M_i is a maximal connected subset of M and $x[0,t]$ is a connected set. Thus if $xt \in M_j$ for some j, $M_i \cup M_j$ is a connected set and M_i and M_j are not disjoint. Thus $xt \notin M_j$ for all j which implies that $xt \notin M$. This contradicts the invariance of M and hence each component of M is invariant. For the converse let each component of M be invariant. Then since $M = \bigcup M_i$ and each component is invariant, it follows that M is invariant. The proofs of the other cases are similar.

Suppose M is invariant, then so is X-M. Consequently cl M and cl (X-M) are invariant. Hence bdy M = cl M $\cap$ cl (X-M) is invariant and int M = X - cl (X-M) is invariant. //

For the following theorem we need to define $\gamma(M)$, $\gamma^+(M)$, and $\gamma^-(M)$ where $M \subset X$. We define these as we would expect; that is $\gamma(M) = \{Mt: t \in R\}$, $\gamma^+(M) = \{Mt: t \in R^+\}$ and similarly for $\gamma^-(M)$.

THEOREM 2.11 For any $x \in X$, the sets $\gamma(x)$, $\gamma^+(x)$, and $\gamma^-(x)$ are, respectively, invariant, positively invariant and negatively invariant. A set $M \subset X$ is invariant, positively invariant, or negatively invariant if and only if, respectively, $\gamma(M) = M$, $\gamma^+(M) = M$, or $\gamma^-(M) = M$.

PROOF Let $xt_1 \in \gamma^-(x)$ for $t_1 \in R^-$ and let $t_2 \in R^-$. Then $xt_1(t_2) = x(t_1 + t_2) = xt_3$ for $t_3 = t_1 + t_2$. Since $t_1, t_2 \in R^-$, we have $t_3 \in R^-$ and $xt_3 \in \gamma^-(x)$. Hence $\gamma^-(x)$ is negatively invariant. The proof of the other cases is entirely analogous. For the second part, let M be negatively invariant. Now $\gamma^-(M) = \{Mt: t \in R^-\}$. In particular for $t = 0$, $Mt = M$ and $M \subset \gamma^-(M)$. Suppose $\gamma^-(M) \not\subset M$. Then there exists a $t_1 \in R^-$ such that $Mt_1 \not\subset M$. Then there exists $x \in M$ such that $xt_1 \notin M$ which contradicts the negative invariance of M. Thus $\gamma^-(M) \subset M$ and $\gamma^-(M) = M$. Conversely, suppose $\gamma^-(M) = M$. Then $\{Mt: t \in R^-\} = M$ and $Mt \subset M$ for all $t \in R^-$. Thus M is negatively invariant. The proofs of the other cases are similar.

The following remarks are useful in understanding this material and prove some useful results.

REMARKS 2.12

1. The sets X and $\emptyset$ are invariant.

PROOF Let $x \in X$. Then $xt \in X$ for all $t \in R$ since $xt \notin \emptyset$. Thus X is invariant. Since $\emptyset = X-X$, it follows that $\emptyset$ is invariant.

2. A set $M \subset X$ is invariant, positively invariant, or negatively invariant if and only if for each $x \in M$, respectively, $\gamma(x) \subset M$, $\gamma^+(x) \subset M$, or $\gamma^-(x) \subset M$.

PROOF Let M be invariant. Then by Theorem 2.11 $\gamma(M) = M$. Thus $\{Mt: t \in R\} = M$. Hence for any $x \in M$, $xt \in M$ for all $t \in R$. But $\{xt: t \in R\} = \gamma(x)$. Thus $\gamma(x) \subset M$. Conversely, suppose $\gamma(x) \subset M$ for each $x \in M$. Then $\{xt: t \in R\} \subset M$ and $xt \in M$ for all $t \in R$ and all $x \in M$. Hence M is invariant. The proofs of the other cases are similar.

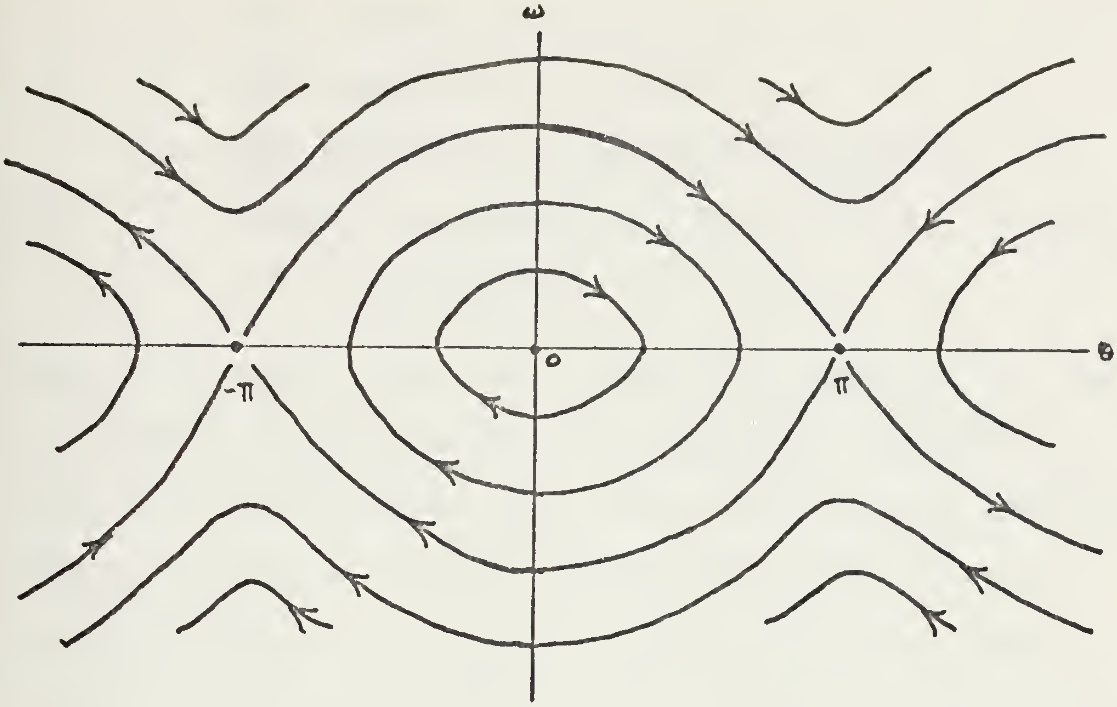
Consider once again Example 2.1. For the points $\binom{n\pi}{0}$ $n = \pm 1, \pm 2, \ldots$ we see that $\dot{\theta} = \omega = 0$ and $\dot{\omega} = -k^2 \sin \theta = 0$. This shows that each $x = \binom{n\pi}{0}$ does not change position with increasing or decreasing time. Such points are called critical or rest points. We now state the formal definition.

DEFINITION 2.13 A point $x \in X$ is said to be a critical point if $x = xt$ for all $t \in R$.

REMARK 2.14 If x is critical and $y \neq x$, then x does not belong to the trajectory $\gamma(y)$.

PROOF Suppose x is critical and $y \neq x$. Then $\{x\} = \gamma(x)$ and since $\gamma(x)$ is invariant, $\{x\}$ is invariant. Thus $X-\{x\}$ is invariant. Since $y \neq x$, $y \in X-\{x\}$ so $\gamma(y) \subset X-\{x\}$. Thus $x \notin \gamma(y)$. //

With this remark, Figure 2.1 should more properly be shown as follows:



The next theorem contains several characterizations of critical points but first the following lemma will be found useful.

LEMMA 2.15 If $x \varepsilon X$ and $x = xt$ for some $t \varepsilon R$, then $x = x(nt)$ for all integers n.

PROOF If $x = xt$, then $x(-t) = xt(-t) = x(t - t) = x$ so we need only consider positive integers. We complete the proof by induction. Clearly if $x = xt$ for some $t \varepsilon R^+$, then $x = x(1t)$. Assume $x = x(nt)$. Then $x = xt(nt) = x(t + nt) = x((n + 1)t)$. //

THEOREM 2.16 Let $x \varepsilon X$. Then the following are equivalent.

1. The point x is critical
2. The singleton $\{x\}$ is $\gamma(x)$

3. The singleton $\{x\}$ is $\gamma^+(x)$
4. The singleton $\{x\}$ is $\gamma^-(x)$
5. The singleton $\{x\}$ is $x[a,b]$ for some $a < b$
6. There is a sequence $\{t_n\}$, $t_n > 0$, $t_n \to 0$ with $x = xt_n$ for each n.

PROOF The equivalence of statements 1. with 2., 3., and 4. are trivial. We next prove the equivalence of statemnts 1. and 6. If x is critical then statement 6 follows trivially since $x = xt$ for all $t \in R$. Conversely, assume there is a sequence $\{t_n\}$, $t_n > 0$, $t_n \to 0$ with $x = xt_n$ for each n and let $t \in R$. If $t = kt_n$ for some integers k and n, then $xt = x(kt_n) = x$ by the lemma. Otherwise since $t_n > 0$ for all integers n, $t/t_n \in R$ and is located between two integers k_n and $k_n + 1$. Thus $k_n < t/t_n < k_n + 1$ and $k_n t_n < t < (k_n + 1)t_n$. Let $\delta = \min\{(k_n + 1)t_n - t, t - k_n t_n\}$. Since $t_n \to 0$, there exists an integer $m > n$ such that $t_m < \delta/2$. As above there exists integers k_m and $k_m + 1$ such that $k_m t_m < t < (k_m + 1)t_m$. Since this interval has length $(k_m + 1)t_m - k_m t_m = t_m < \delta/2$, this interval must be contained inside the previous interval since $\max\{(k_m + 1)t_m - t, t - k_m t_m\} < \delta/2$. Thus we have

$$k_n t_n < k_m t_m < t < (k_m + 1)t_m < (k_n + 1)t_n.$$

Continuing in this manner, we have constructed a sequence $\{k_n t_n\}$ with the property that $k_n t_n \to t$. Now by the

continuity axiom, $x(k_n t_n) \to xt$ and since $x(k_n t_n) = x$ for each n, by the lemma, we have $x = xt$. Thus x is critical.

To prove the equivalence of statements 1. and 5. we first note that if x is critical, then 5. follows trivially. On the other hand, suppose $x = x[a,b]$ for some $a < b$ and let $t \in R$. If $t = kt_1$ for some integer k and t_1 in $[a,b]$, then $xt = x(kt_1) = x$ by the lemma. Otherwise, since $[a,b]$ is closed, there is a sequence $\{t_n\}$ in $[a,b]$ with $t_n \to a$. Consider the sequence $\{t_n - a\}$. Clearly $(t_n - a) > 0$, $(t_n - a) \to 0$ and $x = xa = x(-1a)$ by the lemma. Since $t_n \in [a,b]$, $x = xt_n$. Combining these last two statements we have $x = xa = x(-a) = xt_n(-a) = x(t_n - a)$. Thus the hypotheses of statement 6. are satisfied and x is critical. //

From this theorem we know that if we follow a point of any trajectory in a dynamical system and discover that that point is at rest for any period of time, then the point remains at rest indefinitely.

Continuing to seek motivation from our examples, we again refer to Example 2.3. We see that if x is any point on the trajectory, $x = X(X_0,t)$ for some $t \in R$ but also $x = X(X_0,t + 2n\pi)$ for $n = \pm1, \pm2, \ldots$ Transfering this to our notation of a dynamical system, we have that $x = X_0t = X_0(t + 2n\pi)$. As we would expect, any point with this property is called a periodic point. Formally, we state:

DEFINITION 2.17 A point $x \in X$ is said to be periodic if there is a $T \neq 0$ such that

$$xt = x(t + T) \quad \text{for all } t \in R.$$

A number T for which this holds is called a period of x. If x is a periodic point then both the motion and the trajectory are said to be periodic.

The restriction that $T \neq 0$ is necessary since it is true that $xt = x(t + 0)$ for all t but the point x may not be periodic. It is also noted that any critical point x is periodic since every $T \in R$ is a period for x. The following theorems and remarks prove some of the expected properties of these points.

THEOREM 2.18 If $\{x_n\}$ is a sequence of periodic points with positive periods $T_n \to 0$ and $x_n \to x$, then x is critical.

PROOF As in the proof of Theorem 2.16, given any $t \in R$, there are integers k_n and $k_n + 1$ such that

$$k_nT_n \leq t < (k_n + 1)T_n = k_nT_n + T_n.$$

Since $T_n \to 0$, $k_nT_n \to t$. Then $x_n = x_nT_n$ since x_n is periodic and $x_nT_n = x_n(k_nT_n)$ by lemma 2.15. Combining these last two statements we have $x_n = x_nT_n = x_n(k_nT_n) \to x_nt$. Now

since $x_n \to x$ we have $x = xt$. As $t \in R$ was arbitrary, $x = xt$ for all $t \in R$. Hence x is critical. //

From the definition we can see that a point $x \in X$ is periodic if and only if there is a $T \neq 0$ with $x = xT$. For if x is periodic, we know there exists a $T \neq 0$ such that $xt = x(t + T)$ for all $t \in R$. But in particular, for $t = 0$, we have $x = xT$. On the other hand, if there exists a $T \neq 0$ such that $x = xT$ and if t is any element in R, then $xt = xT(t) = x(T + t) = x(t + T)$. Since t was arbitrary, we see that x is periodic.

In Example 2.3 we noted that $T = 2\pi$ was a period of x and $T = 4\pi$ was a period of x. Clearly in this example, x has an infinite number of periods. We would expect that x would have a smallest non-negative period, which we will call its fundamental period. If there exists an infinite number of periods for a point x in a dynamical system whose fundamental period is 0, then as an immediate consequence of the last theorem, we know that x is critical. This is true since we can certainly form a sequence $\{T_n\}$ of the periods such that $T_n \to 0$ and since the sequence $\{x_n\}$ where $x_n = x$ for all n has the property that $x_n \to x$, the hypotheses of the theorem are fulfilled. The following theorem proves the existence of a fundamental period and also proves why we need only consider positive periods.

THEOREM 2.19 If $x \in X$ is periodic but not critical, then there is a $T > 0$ such that T is the smallest positive period

of x. *Further if* T' *is any other period of* x, *then* $T' = nT$ *for some integer* n.

PROOF Let $P = \{t > 0: t \text{ is a period of } x\}$. $P \neq \emptyset$ since if T is a period of x, $T \neq 0$, then $x = xT$. Thus $x(-T) = xT(-T) = x(T-T) = x$ so $-T$ is also a period of x. Since either T or $-T$ is positive, $P \neq \emptyset$. Now set $T = \inf P$. Such a T exists since P is bounded below by 0 and hence has an infimum. Clearly $T \geq 0$ so suppose $T = 0$. Then there is a sequence $\{t_n\}$ in P with $t_n \to 0$. Since $x = xt_n$ for all n, x must be a critical point by Theorem 2.19. Hence $T > 0$. We see that T is also a period of x since there is a sequence $\{t_n\}$ in P with $t_n \to T$ because $T = \inf P$ and $x = xt_n$ for all n. Thus by the continuity axiom, $x = xt_n \to xT$. Thus T is a period of x and by definition it is the smallest positive period of x. Finally let $t \in R$ be any period of x. If $t \neq nT$ for any integer n, then there exists an integer n such that $nT < t < (n + 1)T$. But since T is a period of x, nT is a period of x by lemma 2.15. Thus we have $x = xt = x(nT)$ which gives $xt(-nT) = x(nT)(-nT) = x(nT-nT) = x$. Then $x = xt(-nT) = x(t-nT)$ which shows that $t-nT$ is also a period of x. But since $0 < t-nT < T$ we have a contradiction since T was the smallest positive period of x. //

REMARK 2.20 If $x \in X$ is periodic, then $\gamma(x)$ and $\gamma^+(x)$ are compact.

PROOF Let x be periodic with positive period T. Then $\gamma(x) = x[0,T]$ and $\gamma^+(x) = x[0,T]$ since for any $t \in [0,T]$, $xt \in x[0,T]$ and for any t with $t > T$, $t = t_1 + kT$ for some integer k and $t_1 \in [0,T]$. Clearly similar results hold for $t < 0$. From Remark 2.6, $x[0,T]$ is compact. Thus $\gamma(x)$ and $\gamma^+(x)$ are compact. //

From this analysis it is now clear that _if any trajectory in a dynamical system ever crosses itself, then the trajectory must be periodic_. For suppose $xt_1 = xt_2$ for $t_1 < t_2$. Then $x = x(t_2 - t_1) = xT$ where $T = t_2 - t_1 > 0$. But this shows that x must be periodic.

III. PROLONGATIONS AND LIMIT SETS

Referring again to Example 2.2, it was noted that in the damped spring case the solution converges to the point $\binom{0}{0}$ as $t \to \infty$. This solution converges in the sense that there is a sequence $\{t_n\}$ with $t_n \in R$ for all n, $t_n \to \infty$, and $xt_n \to \binom{0}{0}$. This point is called a positive limit point. Formally, the definition is stated:

DEFINITION 3.1 Define maps Λ^+, Λ^- from X into 2^X by setting for each $x \in X$,

1. the set $\Lambda^+(x) = \{y \in X$: there is a sequence $\{t_n\}$ in R with $t_n \to +\infty$ and $xt_n \to y\}$,
2. the set $\Lambda^-(x) = \{y \in X$: there is a sequence $\{t_n\}$ in R with $t_n \to -\infty$ and $xt \to y\}$.

For any $x \in X$, the set $\Lambda^+(x)$ is called its positive limit set, and the set $\Lambda^-(x)$ is called its negative limit set.

REMARK 3.2 If $x \in X$ is periodic, then $\Lambda^+(x) = \Lambda^-(x) = \gamma(x)$.

PROOF Let $x \in X$ be periodic. By Remark 2.20, $\gamma(x)$ is compact. Let $y \in \gamma(x)$. Then $y = xt$ for some $t \in R$. Since x is periodic, $y = x(t + T) = xT(t + T) = x(t + 2T)$. Clearly this idea can be extended inductively so consider the sequence $\{t + nT\}$. Since $T > 0$, $t + nT \to +\infty$ and $y = x(t + nT)$ for all n. Thus $x(t + nT) \to y$ and $y \in \Lambda^+(x)$. Hence $\gamma(x) \subset \Lambda^+(x)$. Now let $y \in \Lambda^+(x)$. Then there exists a

sequence $\{t_n\}$ in R with $t_n \to +\infty$ and $xt_n \to y$. For all n, $xt_n \in \gamma(x)$. Hence $\{xt_n\} \subset \gamma(x)$. Since this sequence converges, it must converge to a point in $\gamma(x)$ because $\gamma(x)$ is compact. Thus $y \in \gamma(x)$ and $\Lambda^+(x) \subset \gamma(x)$. Hence $\Lambda^+(x) = \gamma(x)$. The proof for $\Lambda^-(x) = \gamma(x)$ is entirely analogous. //

An immediate consequence of this remark is that if a point x is a critical point, then $\Lambda^+(x) = \Lambda^-(x) = x$. From these two observations, it is an easy task to determine the positive and negative limit sets of Examples 2.1, 2.2, and 2.3., because in Example 2.1 we have critical points, periodic trajectories, and trajectories which clearly do not converge. In Example 2.2, we have either a periodic trajectory for the undamped case or a trajectory whose positive limit set is the origin and negative limit set is empty. Finally in Example 2.3 we have a periodic orbit. However, a point may not be periodic and yet have more than one point in its positive or negative limit set. The following examples help to illustrate this idea.

EXAMPLE 3.2 Consider the differential system defined in R^2 by the following equations in polar coordinates:

$$\frac{dr}{dt} = r(1 - r)$$

$$\frac{d\theta}{dt} = 1.$$

An extensive analysis is not necessary here because all the facts needed can be derived directly from the equations. This system does in fact define a dynamical system [Bhatia, Szego, p. 20]. Since the change in θ is constant, only the change in r affects the system. Clearly for $r = 0$, $\frac{dr}{dt} = 0$ so the origin is a critical point. For $r = 1$, $\frac{dr}{dt} = 0$ so r is again constant with time and the solution describes a periodic trajectory. Suppose $r < 1$. Then $\frac{dr}{dt}$ is positive so r is increasing with time. If $r > 1$, then $\frac{dr}{dt}$ is negative and r is decreasing with time. Such trajectories are shown in the diagram below. It is noted that although $\frac{dr}{dt}$ is always positive for $r < 1$, r will never reach $r = 1$, for as $r \to 1$, $\frac{dr}{dt} \to 0$.

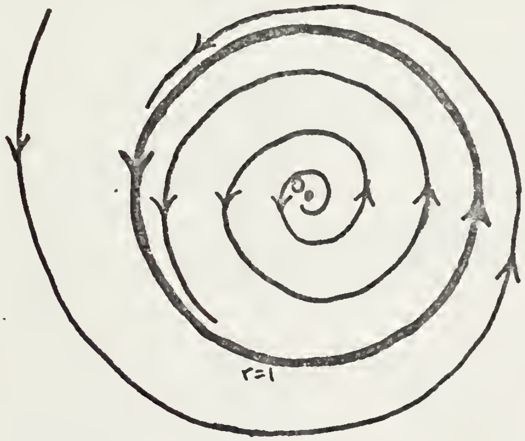


FIGURE 3.2

As this diagram shows, for points p with $0 < r < 1$, $\Lambda^+(p)$ is the unit circle and $\Lambda^-(p)$ is the origin. For points p with $r > 1$, $\Lambda^+(p)$ is the unit circle and $\Lambda^-(p) = \emptyset$. These

facts become clear when it is realized that a sequence $\{t_n\}$ can be found with $t_n \to \infty$ such that xt_n approaches any given point on the unit circle. Since the unit circle is periodic and the origin is a rest point, their positive and negative limit sets are obvious. The direction of the trajectories is counter-clockwise because θ is increasing with time.

EXAMPLE 3.3 Consider the differential system defined in R^2 by the differential equations

$$\dot{x} = f(x,y)$$

$$\dot{y} = g(x,y)$$

where the functions f and g are defined by

$$f(x,y) = \begin{cases} 0 \text{ if } |x| \geq 1 \\ -\dfrac{y(1 - x^2)}{(1 + y^2)(1 - p(x)q(y))} \text{ if } |x| < 1, \end{cases}$$

$$g(x,y) = \begin{cases} 1 \text{ if } x \geq 1 \\ -1 \text{ if } x \leq -1 \\ x \text{ if } |x| < 1. \end{cases}$$

Here the functions p(x) and q(x) are any continuously differentiable functions satisfying the following conditions:

$$0 < p(x) < \frac{1}{2} \qquad \text{if } 0 < x < 1,$$

$$p(x) = 0 \qquad \text{if } x \leq 0,$$

$$p(x) = \frac{1}{2} \qquad \text{if } x \geq 1,$$

$$0 < q(y) < \frac{1}{2} \qquad \text{if } y < 0,$$

$$q(y) = 0 \qquad \text{if } y > 0.$$

This system defines a dynamical system [Bhatia, Szego p. 21] and it is analyzed in the same manner as the preceding example. For $x = y = 0$, $f(x,y) = 0$ and $g(x,y) = 0$ so x and y are unchanging with time. Consequently, the point $\binom{0}{0}$ is a critical point. For $x \geq 1$, $f(x,y) = 0$ and $g(x,y) = 1$. The trajectories of such points are straight lines parallel to the y-axis with direction in the +y direction. A similar analysis shows that for $x \leq -1$, the trajectories are straight lines parallel to the y-axis with direction in the -y direction. Now suppose $0 < x < 1$. Then $g(x,y) > 0$ so y is increasing with time. If $y < 0$, $f(x,y) > 0$ so x is increasing. If $y > 0$, then $f(x,y) < 0$ so x is decreasing. A similar analysis holds for $-1 < x < 0$. Thus the solutions describe spiralling trajectories. The question is do they spiral outward or toward the origin or are they periodic trajectories. Such an analysis from the equations is very involved but the trajectories do in fact spiral outward [Bhatia, Szego, p. 22]. The trajectories are sketched in the diagram below.

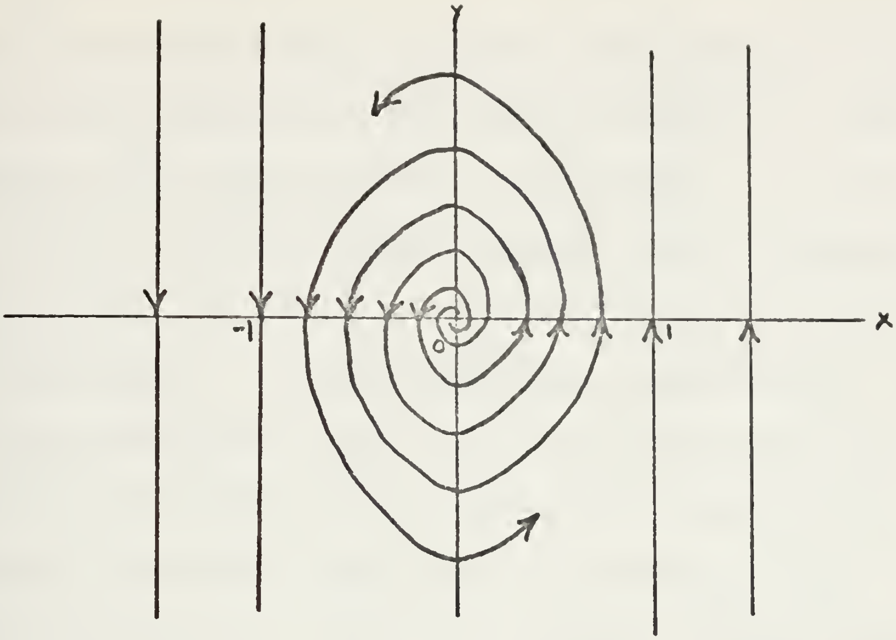


FIGURE 3.3

From the diagram one can see that for any point except the origin in the strip $-1 < x < 1$, its positive limit set is the two straight lines $x = \pm 1$ and its negative limit set is the origin. Since the origin is critical, its positive limit set is itself and for any other point in R^2, its positive and negative limit sets are empty.

Proceeding now with the concept of limit sets, some very important theorems are useful.

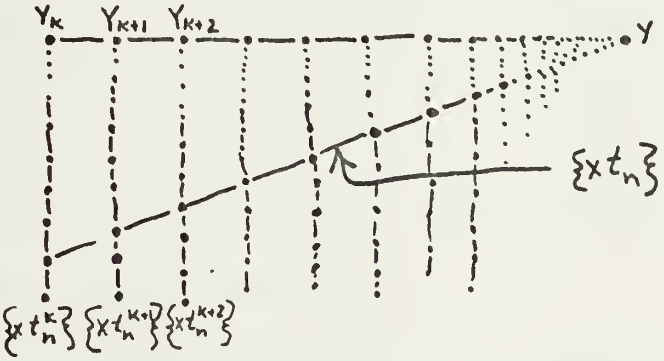
THEOREM 3.4 *For any* $x \in X$,

1. *the sets* $\Lambda^+(x)$ *and* $\Lambda^-(x)$ *are closed and invariant*,

2. *the closure* cl $(\gamma^+(x)) = \gamma^+(x) \cup \Lambda^+(x)$ *and the closure* cl $(\gamma^-(x)) = \gamma^-(x) \cup \Lambda^-(x)$.

PROOF 1. Consider the case of $\Lambda^+(x)$. Let $\{y_n\}$ be a sequence in $\Lambda^+(x)$ with $y_n \to y$. If it can be shown that

$y \in \Lambda^+(x)$, then $\Lambda^+(x)$ is closed. Proceeding in this manner, it is noted that for each positive integer k, there is a sequence $\{t_n^k\}$ in R with $t_n^k \to +\infty$ and $xt_n^k \to y_k$ since each $y_k \in \Lambda^+(x)$. We may assume without loss of generality that $d(y_k, xt_n^k) < 1/k$ and $t_n^k \geq k$ for $n \geq k$ where d is the metric for the space X. These assumptions may be made since the xt_n^k get arbitrarily close to y_k and hence any point in the sequence at distance greater than 1/k from y_k may be removed with the resultant subsequence converging to y_k. The same reasoning applies to $t_n^k \geq k$. Consider now the sequence $\{t_n\}$in R with $t_n = t_n^n$. The significance of this "diagonal" sequence is in consideration of $\{xt_n\}$ and for purposes of clarity, we sketch the latter sequence below:



Then $t_n \to +\infty$ and the claim is that $xt_n \to y$. To see this, observe that

$$d(y, xt_n) \leq d(y, y_n) + d(y_n, xt_n) \leq d(y, y_n) + 1/n.$$

Since 1/n and $d(y, y_n)$ tend to zero, this implies that $d(y, xt_n) \to 0$. Consequently, $xt_n \to y$ and $y \in \Lambda^+(x)$. Thus $\Lambda^+(x)$ is closed. To see that $\Lambda^+(x)$ is invariant, let

$y \in \Lambda^+(x)$ and $t \in R$ be arbitrary. There is a sequence $\{t_n\}$ in R with $t_n \to +\infty$ and $xt_n \to y$. Then by the continuity axiom, $xt_n(t) \to yt$. Since $xt_n(t) = x(t_n + t)$ and $(t_n + t) \to +\infty$ this implies that $yt \in \Lambda^+(x)$. Thus $\Lambda^+(x)$ is invariant.

To prove the second part, it is recalled that $\gamma^+(x) = xR^+$. By the definition of $\Lambda^+(x)$ it follows that $\mathrm{cl}\ (\gamma^+(x)) \supset \gamma^+(x) \cup \Lambda^+(x)$. Now let $y \in \mathrm{cl}\ (\gamma^+(x))$. Then there is a sequence $\{y_n\}$ in $\gamma^+(x)$ such that $y_n \to y$. Since this sequence is in $\gamma^+(x)$ it follows that $y_n = xt_n$ for some $t_n \in R$. Either the sequence $\{t_n\}$ has the property that $t_n \to +\infty$, in which case $y \in \Lambda^+(x)$, or by the Bolzano-Weierstrass theorem there is a subsequence $\{t_{n_k}\}$ with $t_{n_k} \to t \in R^+$. But then $xt_{n_k} \to xt$ which is in $\gamma^+(x)$, and since also $xt_{n_k} \to y$, it follows that $y = xt \in \gamma^+(x)$. Thus $\mathrm{cl}\ (\gamma^+(x)) \subset \gamma^+(x) \cup \Lambda^+(x)$. The proofs for the negative case are entirely similar. //

The properties of limit sets just proved will be found to be very useful in further work.

THEOREM 3.5 For $x,y \in X$, if $y \in \gamma^+(x)$, then $\Lambda^+(y) = \Lambda^+(x)$.

PROOF Let $y \in \gamma^+(x)$. Then there is a $t \in R$ such that $y = xt$. Now let $u \in \Lambda^+(y)$. Then there exists a sequence $\{t_m\}$ in R with $t_m \to +\infty$ such that $yt_m \to u$. Consider the

sequence $\{t + t_m\}$ in R. Clearly $t + t_m \to +\infty$. Then $yt_m \to u$ implies $yt_m = xt(t_m) = x(t + t_m) \to u$. Thus $u \in \Lambda^+(x)$ and hence $\Lambda^+(y) \subset \Lambda^+(x)$.

On the other hand, let $z \in \Lambda^+(x)$. Then there is a sequence $\{t_m\}$ in R with $t_m \to +\infty$ and $xt_m \to z$. Consider the sequence $\{t_m - t\}$ in R. Clearly $(t_m - t) \to \infty$ since $t_m \to \infty$. Then $xt_n \to z$ implies $xt_n = x(t + t_m - t) = xt(t_m - t) = y(t_m - t) \to z$. Thus $z \in \Lambda^+(y)$ and $\Lambda^+(x) \subset \Lambda^+(y)$.

REMARK 3.6 Let $x,y \in X$ such that x is not periodic. Then the set $\bigcap\{\gamma^+(y): y \in \gamma^+(x)\} = \emptyset$.

PROOF Since $y \in \gamma^+(x)$, there is a $t \in R^+$ such that $y = xt$. Now suppose $s \in \bigcap\{\gamma^+(y): y \in \gamma^+(x)\}$. Then $s \in \gamma^+(y)$ which implies that $s = yt_1$ for some $t_1 \in R^+$. But since $y = xt$, it follows that $s = xt(t_1) = x(t + t_1)$. Let $t_2 \in R$ such that $t_2 > t + t_1$. Then for $y = xt_2$, $s \notin \gamma^+(y)$. Thus $s \notin \bigcap\{\gamma^+(y): y \in \gamma^+(x)\}$ and we have a contradiction. Hence $\bigcap\{\gamma^+(y): y \in \gamma^+(x)\} = \emptyset$. //

REMARK 3.7 For any $x \in X$, $\Lambda^+(x) = \bigcap\{\mathrm{cl}\ (\gamma^+(y)): y \in \gamma^+(x)\} = \bigcap\{\gamma^+(xn): n \text{ is an integer}\}$.

PROOF If x is periodic, the proof is trivial since $\mathrm{cl}\ (\gamma^+(y)) = \Lambda^+(x)$ for all y. So we assume x is not periodic.

Let $\{t_n\}$ be any sequence in R such that $t_n \to +\infty$. Without loss of generality it may be assumed that $t_n > 0$ for all n since any points in the sequence for which this does not hold may be removed from the sequence without affecting

$t_n \to +\infty$. Since the intersection in the statement is taken over all possible y's in $\gamma^+(x)$, set $y_n = xt_n$ where each $t_n \in \{t_n\}$. Now $\gamma^+(y_n) = \{y_nt: t \in R^+\} = \{x(t_n + t): t \in R^+\}$. The claim is that $\bigcap \{\mathrm{cl}\ (\gamma^+(y))\} = \bigcap \{\mathrm{cl}\ (\gamma^+(y_n))\}$.

By Theorem 3.4, $\bigcap \{\mathrm{cl}\ (\gamma^+(y))\} = \bigcap \{\gamma^+(y) \cup \Lambda^+(y)\} = \bigcap \{\gamma^+(y)\} \cup \bigcap \{\Lambda^+(y)\}$ and
$\bigcap \{\mathrm{cl}\ (\gamma^+(y_n))\} = \bigcap \{\gamma^+(y_n) \cup \Lambda^+(y_n)\} = \bigcap \{\gamma^+(y_n)\} \cup \bigcap \{\Lambda^+(y_n)\}$.
By Remark 3.6, $\bigcap \{\gamma^+(y)\} = \emptyset = \bigcap \{\gamma^+(y_n)\}$ and by Theorem 3.5, $\Lambda^+(y) = \Lambda^+(x) = \Lambda^+(y_n)$. Thus
$\bigcap \{\mathrm{cl}\ (\gamma^+(y))\} = \bigcap \{\Lambda^+(y)\} = \Lambda^+(x)$ and
$\bigcap \{\mathrm{cl}\ (\gamma^+(y_n))\} = \bigcap \{\Lambda^+(y_n)\} = \Lambda^+(x)$. Then, as claimed,
$\bigcap \{\mathrm{cl}\ (\gamma^+(y))\} = \bigcap \{\mathrm{cl}\ (\gamma^+(y_n))\}$ and as shown above,
$\bigcap \{\mathrm{cl}\ (\gamma^+(y)): y \in \gamma^+(x)\} = \Lambda^+(x)$. To prove the second part, let the sequence $\{t_n\} \equiv \{n\}$. Then the proof is identical. //

In the following proof and elsewhere in this thesis, the notation $S(M,\varepsilon)$ and $S[M,\varepsilon]$ will be used for the sets $\{x: d(x,M) < \varepsilon\}$, and $\{x: d(x,M) \leq \varepsilon\}$ respectively where $M \subset X$, $\varepsilon \in R$ and d is the metric for the space X. In the case where $M = \{x\}$, the notation will be shortened to $S(x,\varepsilon)$ and $S[x,\varepsilon]$.

THEOREM 3.8 *Let* $x \in X$. *If* $\mathrm{cl}\ (\gamma^+(x))$ *is compact, then* $\Lambda^+(x)$ *is a non-empty compact and connected set.*

PROOF Let $\mathrm{cl}\ (\gamma^+(x))$ be compact and let $\{t_n\}$ be any sequence in R^+ such that $t_n \to +\infty$. Now consider the sequence

$\{xt_n\}$. Since $xt_n \in$ cl $(\gamma^+(x))$ for all n and cl $(\gamma^+(x))$ is compact, this sequence has a convergent subsequence $\{xt_k\}$which converges to a point $y \in$ cl $(\gamma^+(x))$. Thus since $t_k \to +\infty$ and $xt_k \to y$, we have $\Lambda^+(x) \neq \emptyset$ since $y \in \Lambda^+(x)$. Now cl $(\gamma^+(x)) = \gamma^+(x) \cup \Lambda^+(x)$. By Theorem 3.4, $\Lambda^+(x)$ is a closed set and since it is a closed subset of a compact set, $\Lambda^+(x)$ is itself compact.

The proof of connectedness is by contradiction. Suppose $\Lambda^+(x)$ is compact but not connected. Then $\Lambda^+(x) = P \cup Q$ where P,Q are non-empty, closed disjoint sets. Since cl $(\gamma^+(x))$ is compact, it is locally compact and hence there exists an $\varepsilon > 0$ such that $S[P,\varepsilon]$ and $S[Q,\varepsilon]$ are compact and disjoint. To see that $S[P,\varepsilon]$ is compact, it is noted that in a locally compact space, every point has at least one compact neighborhood. Taking the union of all these compact neighborhoods, yields a compact neighborhood of the set P. Then there is an $\varepsilon > 0$ such that $S[P,\varepsilon]$ is a closed subset of this neighborhood and hence $S[P,\varepsilon]$ is compact. The same reasoning applies to $S[Q,\varepsilon]$. Since a metric space is normal, there exist disjoint open sets containing P and Q respectively. Hence for a proper choice of ε, the sets $S[P,\varepsilon]$ and $S[Q,\varepsilon]$ are disjoint and compact.

Now let $y \in P$ and $z \in Q$. Then there exist sequences $\{t_n\}$ and $\{\tau_n\}$ with $t_n \to +\infty$ and $\tau_n \to +\infty$ such that $xt_n \to y$ and $x\tau_n \to z$. Without loss of generality it may be assumed that $xt_n \in S(P,\varepsilon)$, $x\tau_n \in S(Q,\varepsilon)$ and $\tau_n - t_n > 0$ for all n.

Since the trajectory segments $x[t_n,\tau_n]$, $n = 1, 2, \ldots$ are compact connected sets, they clearly intersect $S(P,\varepsilon)$ and $S(Q,\varepsilon)$. Thus in particular, there is a sequence $\{T_n\}$ with $t_n < T_n < \tau_n$ such that $xT_n \in H(P,\varepsilon)$ where $H(P,\varepsilon) = \{x: d(x,P) = \varepsilon\}$. Since $H(P,\varepsilon)$ is a closed subset of $S[P,\varepsilon]$, $H(P,\varepsilon)$ is compact. Since $\{xT_n\}$ is a sequence in $H(P,\varepsilon)$, it has a convergent subsequence $\{xT_m\}$ which converges to a point y' and because $T_m \to +\infty$, it follows that $y' \in \Lambda^+(x)$. But $y' \not\in P \cup Q$ which contradicts the assumption that $\Lambda^+(x) = P \cup Q$. Hence $\Lambda^+(x)$ is connected. //

Since only the local compactness of cl $(\gamma^+(x))$ was used in the proof, an immediate corollary to this theorem is that if the space X is locally compact, then $\Lambda^+(x)$ is connected whenever it is compact.

The converse of the previous theorem holds if the space X is locally compact. However, the proof relies on the one-point compactification of X. The fact that the one-point compactification is an extension of a dynamical system to another dynamical system is proved in the following lemma.

LEMMA Given a dynamical system (X,R,π) on a locally compact space X, let $X' = X \cup \{\omega\}$ be the one-point compactification of X and define $\pi': X' \times R \to X'$ by

$$\pi'(x,t) = \begin{cases} \pi(x,t) & \text{if } x \in X, \quad t \in R \\ \omega & \text{if } x = \omega, \quad t \in R \end{cases}$$

Then (X',R,π') is a dynamical system on X'.

PROOF First, by definition of π',

$$\pi'(x,0) = \begin{cases} \pi(x,0) = x & \text{if } x \in X \\ \omega & \text{if } x = \omega . \end{cases}$$

Thus $\pi'(x,0) = x$ for all $x \in X'$.

Secondly, if $x \in X$, then $\pi'(\pi'(x,t_1),t_2) = \pi'(x,t_1 + t_2)$ follows trivially so suppose $x = \omega$. Then $\pi'(\pi'(\omega,t_1),t_2) = \pi'(\omega,t_2) = \omega = \pi'(\omega,t_1 + t_2)$. Thus $\pi'(\pi'(x,t_1),t_2) = \pi'(x,t_1 + t_2)$ for all $x \in X'$. It remains to show that π' is continuous. To see this, we will utilize the characterization that if the inverse image of an open set is open, then the map is continuous.

So let U be any open set in X'. If $\omega \notin U$, then U is open in X and thus $(\pi')^{-1}(U) = \pi^{-1}(U)$. Now $\pi^{-1}(U)$ is open because π is continuous and hence $(\pi')^{-1}(U)$ is open. If $\omega \in U$, then X-U is compact and since $\omega \notin X'-U$, $X'-U = X-U$ and hence $(\pi')^{-1}(X'-U) = (\pi')^{-1}(X-U) = \pi^{-1}(X-U)$. Now since π is continuous, $\pi^{-1}(X-U)$ is closed and hence $(\pi')^{-1}(X'-U)$ is closed so $(\pi')^{-1}(U)$ is open. Thus the inverse image of every open set is open and hence π' is continuous. //

THEOREM 3.9 If the space X is locally compact, then for any $x \in X$, $cl(\gamma^+(x))$ is compact whenever $\Lambda^+(x)$ is non-empty and compact.

PROOF Let $\Lambda^+(x)$ be non-empty and compact. Consider the extension of the dynamical system to the one-point compactification $X' = X \cup \{\omega\}$. Certainly, for any $x \in X$, $\tilde{\Lambda}^+(x) = \Lambda^+(x) \cup \{\omega\}$ or $\tilde{\Lambda}^+(x) = \Lambda^+(x)$ where $\tilde{\Lambda}^+(x)$ is the positive limit set of x in X'. However since $\Lambda^+(x)$ is compact, it follows that $\tilde{\Lambda}^+(x) = \Lambda^+(x)$ holds because $\tilde{\Lambda}^+(x) = cl_{X'}(\Lambda^+(x)) = cl_X(\Lambda^+(x)) = \Lambda^+(x)$. Also we have that for any $x \in X$, $\tilde{\gamma}^+(x) = \gamma^+(x)$ since $\pi'(x,t) = \pi(x,t)$. Thus we have

$$cl_{X'}(\tilde{\gamma}^+(x)) = \tilde{\gamma}^+(x) \cup \tilde{\Lambda}^+(x) = \gamma^+(x) \cup \Lambda^+(x) = cl_X(\gamma^+(x))$$

and since $cl_{X'}(\tilde{\gamma}^+(x))$ is a closed subset of a compact space, it is compact. Thus $cl(\gamma^+(x))$ is compact. //

THEOREM 3.10 If $cl(\gamma^+(x))$ is compact, then $d(xt,\Lambda^+(x)) \to 0$ as $t \to +\infty$.

PROOF Let $cl(\gamma^+(x))$ be compact and suppose $d(xt,\Lambda^+(x)) \not\to 0$ as $t \to +\infty$. Then there exists a sequence $\{t_n\}$ in R such that $t_n \to +\infty$ and $d(xt_n,\Lambda^+(x)) > \varepsilon$ for all $n > N$ where N is an integer and $\varepsilon > 0$. But since $cl\ (\gamma^+(x))$ is compact, the sequence $\{xt_n\}$ has a convergent subsequence $\{xt_k\}$ which converges to a point $p \in cl\ (\gamma^+(x))$. Now since $t_n \to +\infty$, it follows that $t_k \to +\infty$ and since $xt_k \to p$, we have that $p \in \Lambda^+(x)$. Also since $xt_k \to p$, this sequence gets arbitrarily close to p and hence $d(xt_k,p) < \varepsilon$ for k sufficiently large.

Hence we have $d(xt_k, \Lambda^+(x)) < \varepsilon$ for k sufficiently large but this is a contradiction and the theorem is proved. //

The study of limit sets can be very useful in predicting the behavior of a system. If the system describes a physical problem and the trajectories are sketched, it is a relatively simple task to discover the limit sets and hence predict the future behavior of the system. For example, consider again the pendulum problem, Example 2.1. If it is known that the pendulum is started at the equilibrium position and given an initial angular velocity of $\omega = 2k$, then one would predict that the pendulum would approach the position where $\theta = \pi$ and would never change direction to swing back. This result would be known without having someone watch the pendulum to actually see what happens. It is true that this is a very elementary problem, but the ideas involved have broader applications to problems of much greater difficulty.

It is important that the reader have a firm understanding and familiarity with the symbols and sets thus far defined because the following additional sets and symbols become quite important in the next two sections of this thesis.

DEFINITION 3.11 For each $x \in X$, define the following subsets D^+, D^-, J^+, J^- of X,

1. the set $D^+(x) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R such that $x_n \to x$ and $x_n t_n \to y\}$,

2. the set $D^-(x) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^- such that $x_n \to x$ and $x_n t_n \to y\}$,

3. the set $J^+(x) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$in R^+ such that $x_n \to x$, $t_n \to +\infty$, and $x_n t_n \to y\}$,

4. the set $J^-(x) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$in R^- such that $x_n \to x$, $t_n \to -\infty$, and $x_n t_n \to y\}$.

For any $x \in X$, the set $D^+(x)$ is the first positive prolongation of x and the set $D^-(x)$ is the first negative prolongation of x.

The sets $J^+(x)$ and $J^-(x)$ are called, respectively, the first positive and the first negative prolongational limit set of x.

These sets are called "first" because higher-order prolongations can be defined. However, in this thesis, such higher-order prolongations will not be discussed. Consequently, the word "first" will be suppressed.

From the definitions it is clear that any $x \in X$, $D^+(x) \supset \gamma^+(x)$, $D^-(x) \supset \gamma^-(x)$, $J^+(x) \supset \Lambda^+(x)$, and $J^-(x) \supset \Lambda^-(x)$. To achieve an intuitive feeling for these sets and to see that the inclusions just noted may be proper, we present the following example.

EXAMPLE 3.12 Consider a predator-prey problem where it is assumed that a single predator population feeds on a single prey population. Let x denote the size of the prey

population and let y denote the size of the predator population. Then this problem can be expressed by the system

$$\frac{dx}{dt} = ax - bxy, \quad x \geq 0,$$

$$\frac{dy}{dt} = cxy - dy, \quad y \geq 0,$$

where a,b,c, and d are all non-negative constants.

These equations support our intuition since we would expect that any change in the size of the predator population depends not only on the size of the prey population but also on the size of the predator population. For example if the prey population is small and the predator population is large, we would expect that the predator population would decrease. This expectation is confirmed by the equations.

Rather than solving this system in terms of t, it will be analyzed directly from the differential equations. Clearly for $x = 0 = y$, $\frac{dx}{dt} = 0 = \frac{dy}{dt}$ and the point $\binom{0}{0}$ is a critical point. One can also see from the equations that the point $\binom{d/c}{a/b}$ is a critical point. For any point $p = \binom{x}{0}$, we have $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} = 0$. Thus its associated trajectory lies on the x-axis. Physically, this is the case when there is no predator population so the prey population increases. For any point $p = \binom{0}{y}$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} < 0$ so its trajectory lies on the y-axis with direction

in the negative y-direction. This is the case when there is no prey population so the size of the predator population decreases. For small x and large y, $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0$. Thus both x and y are decreasing and x continues to do so until $y = a/b$. At this point $\frac{dx}{dt} = 0$ and as y continues to decrease, x begins to increase. When x reaches a value of d/c, $\frac{dy}{dt} = 0$ and as x continues to increase, y begins to increase. The physical situation described by this case is quite clear. The trajectories for this system are sketched below.

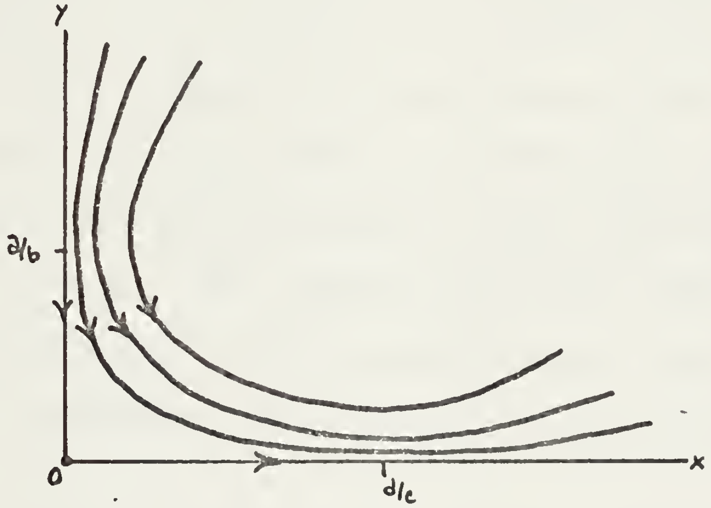


FIGURE 3.12

For any point $p = \binom{0}{y}$ we have $\Lambda^+(p) = \binom{0}{0}$ but $J^+(p) = \{\binom{x}{0} : x \geq 0\}$. The fact that $\Lambda^+(p) = \binom{0}{0}$ means that if there is no prey population, the predator population will eventually decrease to nothing. The fact that $J^+(p) = \binom{x}{0}$ means that even though the prey population may decrease considerably so that the predator population decreases to nothing, there will be enough of the prey population left

to eventually increase its size to any amount. We also notice from this diagram that for any point $p = \binom{0}{y}$, $\Lambda^+(p) = \binom{0}{0}$ but $J^+(p) = \{\binom{x}{0} : x \geq 0\}$. Also $D^+(p) = \gamma^+(p) \cup \{\binom{x}{0}\}$, $D^-(p) = \gamma^-(p)$ and $J^-(p) = \emptyset$. For any point $p = \binom{x}{0}$, $D^+(p) = \gamma^+(p)$, $J^+(p) = \emptyset$, $D^-(p) = \gamma^-(p) \cup \{\binom{0}{y}\}$, and $J^-(p) = \{\binom{0}{y}\}$. For any other point p, $D^+(p) = \gamma^+(p)$, $D^-(p) = \gamma^-(p)$, $J^+(p) = \emptyset = J^-(p)$. It may be argued here that we cannot form infinite sequences since "size" is not a continuous function. However, in an actual experiment, the following values for the constants a,b,c,d were observed: $a \sim 1$, $b \sim .01$, $c \sim 3 \times 10^{-8}$, and $d \sim 0.45$. These values were obtained by observing paramecia feeding on yeast cells. The value of y for which $\frac{dx}{dt}$ changes sign is then at $y \sim 100$ and the value of x for which $\frac{dy}{dt}$ changes sign is at $x \sim 1.5 \times 10^6$. These values are large enough to assume continuity. Also any mathematical model idealizes the physical situation: the question is how "close" [Bailey, p. 245, 246].

Now we consider some basic properties and important theorems about these limit sets.

THEOREM 3.13 For any $x \in X$,

1. the set $D^+(x)$ is closed and positively invariant,
2. the set $J^+(x)$ is closed and invariant,
3. the equality $D^+(x) = \gamma^+(x) \cup J^+(x)$ holds.

Analogous results hold for $D^-(x)$ and $J^-(x)$.

PROOF Statement 1 will follow from statements 2 and 3 so statement 2 is proved first. This proof parallels the proof of Theorem 3.4. Let $\{y_n\}$ be a sequence in $J^+(x)$ with $y_n \to y$. For each integer k, there are sequences $\{x_n^k\}$ in X and $\{t_n^k\}$ in R^+ with $x_n^k \to x$, $t_n^k \to +\infty$, and $x_n^k t_n^k \to y_k$. Without loss of generality, it may be assumed that $t_n^k > k$, $d(x_n^k, x) \leq 1/k$, and $d(x_n^k t_n^k, y_k) \leq 1/k$ for $n \geq k$. Now consider the sequences $\{x_n^n\}$, $\{t_n^n\}$. Clearly $x_n^n \to x$, $t_n^n \to +\infty$, and $x_n^n t_n^n \to y$. This is true since

$d(x_n^n t_n^n, y) \leq d(x_n^n t_n^n, y_n) + d(y_n, y) \leq 1/n + d(y_n, y)$. Thus

$y \in J^+(x)$ and $J^+(x)$ is closed.

To see that $J^+(x)$ is invariant, let $y \in J^+(x)$ and $t \in R$. There is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+ such that $t_n \to +\infty$, $x_n \to x$, and $x_n t_n \to y$. Now consider the sequence $\{t_n + t\}$. Clearly $t_n + t \to +\infty$, and $x_n(t_n + t) = x_n t_n(t) \to yt$. Since $x_n \to x$, it follows that $yt \in J^+(x)$. Since $t \in R$ was arbitrary, it follows that $J^+(x)$ is invariant.

Consider statement 3. Observe that $D^+(x) \supset \gamma^+(x) \cup J^+(x)$ always holds from the definition. So let $y \in D^+(x)$. Then there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+ with $x_n \to x$, $x_n t_n \to y$. It may be assumed that either $t_n \to t \in R^+$ or $t_n \to +\infty$, if necessary by taking subsequences. In the first case $x_n t_n \to xt$ by the continuity axiom. Hence, $xt = y \in \gamma^+(x)$ as $t \in R^+$. In the second case

$y \in J^+(x)$ by definition. Thus $y \in \gamma^+(x) \cup J^+(x)$ and $D^+(x) \subset \gamma^+(x) \cup J^+(x)$. Therefore, $D^+(x) = \gamma^+(x) \cup J^+(x)$.

To see statement 1, recall that $\mathrm{cl}\ (\gamma^+(x)) = \gamma^+(x) \cup \Lambda^+(x)$. But it has been shown that $\Lambda^+(x) \subset J^+(x)$. Hence $D^+(x) = \mathrm{cl}\ (\gamma^+(x)) \cup J^+(x)$, because $J^+(x) \supset \Lambda^+(x)$. Since both these sets are closed and positively invariant, it follows that $D^+(x)$ is closed and positively invariant. //

The following theorem will also be found useful in subsequent sections.

THEOREM 3.14 Let X be locally compact. Then $\Lambda^+(x) \neq \emptyset$ whenever $J^+(x)$ is non-empty and compact.

PROOF Suppose $\Lambda^+(x) = \emptyset$. Then $\mathrm{cl}\ (\gamma^+(x)) = \gamma^+(x) \cup \Lambda^+(x) = \gamma^+(x)$ since $\Lambda^+(x) = \emptyset$. Hence $\gamma^+(x)$ is closed. It is claimed that $\gamma^+(x)$ is disjoint from $J^+(x)$ for if $\gamma^+(x) \cap J^+(x) \neq \emptyset$, then by the invariance of $J^+(x)$, $\gamma^+(x) \subset J^+(x)$. Since $J^+(x)$ is compact, for any sequence $\{t_n\}$ in R^+ such that $t_n \to +\infty$, the sequence $\{xt_n\}$ must have a convergent subsequence. It follows that the point to which that subsequence converges is a point in $\Lambda^+(x)$ and hence $\Lambda^+(x) \neq \emptyset$. Thus $\gamma^+(x) \cap J^+(x) = \emptyset$ as claimed. Since $J^+(x)$ is non-empty and compact, $d(\gamma^+(x), J^+(x)) > 0$. Thus since X is locally compact, there is a $\delta > 0$ such that $S[J^+(x), \delta]$ is compact and disjoint from $\gamma^+(x)$. Now choose any $y \in J^+(x)$. There is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+ such that $x_n \to x$,

$t_n \to +\infty$, and $x_n t_n \to y$. It may be assumed that $x \not\in S[J^+(x),\delta]$, $x_n t_n \in S[J^+(x),\delta]$ for all n. Then the trajectory segments $x_n[0,t_n]$ intersect $H(J^+(x),\delta)$ where $H(J^+(x),\delta) = \{z \in X : d(J^+(x),z) = \delta\}$. Therefore there is a sequence $\{\tau_n\}$, $0 < \tau_n \leq t_n$, such that $x_n\tau_n \in H(J^+(x),\delta)$. Since $H(J^+(x),\delta)$ is compact, it may be assumed that $x_n\tau_n \to z \in H(J^+(x),\delta)$. By taking subsequences if necessary, it may be assumed that either $\tau_n \to t \in R^+$ or $\tau_n \to +\infty$. If $\tau_n \to t \in R^+$, then by the continuity axiom $x_n\tau_n \to xt = z$ so $z \in \gamma^+(x)$ which contradicts $\gamma^+(x) \cap S[J^+(x),\delta] = \emptyset$. If $\tau_n \to +\infty$, then $z \in J^+(x)$, but this contradicts $z \in H(J^+(x),\delta)$ as $J^+(x) \cap H(J^+(x),\delta) = \emptyset$. Thus the original assumption that $\Lambda^+(x) = \emptyset$ is untenable and hence $\Lambda^+(x) \neq \emptyset$. //

THEOREM 3.15 Let X be locally compact. Then $J^+(x)$ is non-empty and compact if and only if $D^+(x)$ is compact.

PROOF Let $J^+(x)$ be non-empty and compact. Then by the preceding theorem, $\Lambda^+(x)$ is non-empty and, since it is a closed subset of $J^+(x)$, it is compact. But then $\mathrm{cl}\,(\gamma^+(x))$ is compact by Theorem 3.9. Hence $D^+(x) = \gamma^+(x) \cup J^+(x) = \mathrm{cl}\,(\gamma^+(x)) \cup J^+(x)$ so that $D^+(x)$ is compact. On the other hand, suppose $D^+(x)$ is compact and assume $J^+(x) = \emptyset$. But $J^+(x) = \emptyset$ implies $\gamma^+(x)$ is compact. Then $\gamma^+(x) = \mathrm{cl}\,(\gamma^+(x))$ so $\mathrm{cl}\,(\gamma^+(x))$ is compact but this implies $\Lambda^+(x) \neq \emptyset$ by Theorem 3.8 and the assumption

that $J^+(x) = \emptyset$ is contradicted. Thus $J^+(x) \neq \emptyset$ and since $J^+(x)$ is a closed subset of $D^+(x)$, it follows that $J^+(x)$ is compact. //

Before proceeding to the next theorem, the set $D^+(M)$ must be defined where $M \subset X$. Thus $D^+(M) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+ such that $x_n \to x \in M$, and $x_n t_n \to y\}$.

THEOREM 3.16 For any compact set $M \subset X$, the set $D^+(M)$ is closed and positively invariant.

PROOF Let $\{y_n\}$ be any sequence in $D^+(M)$ such that $y_n \to y$. Then for each y_k there exist sequences $\{x_n^k\}$ in X and $\{t_n^k\}$ in R^+ such that $x_n^k \to x_k$ and $x_n^k t_n^k \to y_k$ where $x_k \in M$. Without loss of generality it may be assumed that $d(x_n^k t_n^k, y_k) < 1/k$. Now consider the sequences $\{x_n^n\}$ in M and $\{t_n^n\}$ in R^+. Then $x_n^n t_n^n \to y_n$ and since M is compact, it may be assumed that $x_n^n \to x \in M$. Thus $x_n^n t_n^n \to y$ since

$d(x_n^n t_n^n, y) \leq d(x_n^n t_n^n, y_n) + d(y_n, y) \leq 1/n + d(y_n, y)$. Then by definition, $y \in D^+(M)$ and hence $D^+(M)$ is closed. To see that $D^+(M)$ is positively invariant let $y \in D^+(M)$ and let $t \in R^+$ be arbitrary. Now $y \in D^+(M)$ implies there exist sequences $\{x_n\}$ in X and $\{t_n\}$ in R^+ such that $x_n \to x \in M$ and $x_n t_n \to y$. Now consider the sequence $\{t_n + t\}$ in R^+. Since $x_n t_n \to y$, it follows that $x_n t_n(t) = x_n(t_n + t) \to yt$. Thus $yt \in D^+(M)$. Since $t \in R$ was arbitrary it follows that $D^+(M)$ is invariant. //

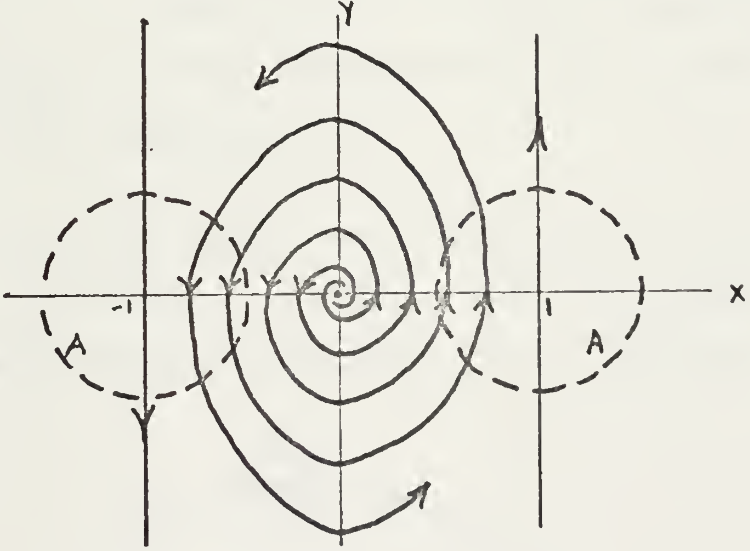
We conclude this section with an important theorem.

THEOREM 3.17 Let x, y in X. Then $y \in J^+(x)$ if and only if $x \in J^-(y)$.

PROOF Let $y \in J^+(x)$. Then there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R such that $x_n \to x$, $t_n \to +\infty$, and $x_n t_n \to y$. Set $x_n t_n = y_n$, and $\tau_n = -t_n$. Then $y_n \to y$, $\tau_n \to -\infty$, and $y_n \tau_n = x_n t_n(\tau_n) = x_n(t_n - t_n) = x_n \to x$. Consequently $x \in J^-(y)$. The converse holds in the same manner. //

IV. RECURSIVE AND DISPERSIVE CONCEPTS

Using the sets defined in the last section, we continue with our analysis of dynamical systems by studying recursive and dispersive concepts. To motivate the idea of recursiveness, consider again Example 3.3. Suppose we consider a set A to be the union of two circles centered respectively at the points $\begin{pmatrix}-1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$ with radius r where $0 < r < 1$. In this example, all trajectories in the strip $-1 < x < 1$ spiral outward from the origin with the lines $x = -1$ and $x = 1$ as their positive limit set. Thus for any point in the strip $-1 < x < 1$, its trajectory will eventually intersect A. The trajectory will pass out of A again but it nevertheless will return some time later and repeat the process. This is shown in the following diagram.



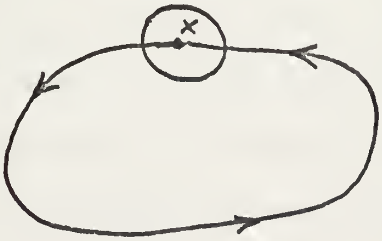
The set A is said to be positively recursive with respect to the trajectory. We now state the formal definition of recursiveness.

DEFINITION 4.1 A set $A \subset X$ is said to be positively recursive with respect to a set $B \subset X$ if for each $T \in R$ there is a $t > T$ and an $x \in B$ such that $xt \in A$. Negative recursiveness may be defined by using the inequality $t < T$. A set A is self positively recursive whenever it is positively recursive with respect to itself.

The simplest example of a self recursive set is the set $\{x\}$ where x is a periodic point. By placing restrictions on the system by demanding that the entering of points from B into the set A happen with some regularity, other concepts such as almost periodicity can be discussed although such is not our purpose here. We now define some concepts and establish properties in connection with the concept of recursiveness.

DEFINITION 4.2 A point $x \in X$ is said to be positively Poisson stable if every neighborhood of x is positively recursive with respect to $\{x\}$.

This definition means that the trajectory through the point x eventually intersects every neighborhood of x. The following diagram illustrates such a situation.



To give a better understanding and some characterizations of positively Poisson stable points, the following theorem is presented.

THEOREM 4.3 *Let* $x \in X$. *Then the following are equivalent.*

1. *The point* x *is positively Poisson stable.*
2. *Given a neighborhood* U *of* x *and a* $T > 0$, *we have* $xt \in U$ *for some* $t > T$,
3. *The point* x *is contained in its positive limit set* $\Lambda^+(x)$,
4. *The set* $\text{cl}\ (\gamma^+(x)) = \Lambda^+(x)$,
5. *The set* $\gamma(x) \subset \Lambda^+(x)$,
6. *For every* $\varepsilon > 0$ *there is a* $t \geq 1$ *such that* $xt \in S(x,\varepsilon)$.

PROOF The equivalence of statements 1 and 2 follows from the definition and our previous discussion. To see the equivalence of statements 2 and 3, consider a sequence $\{T_n\}$ in R^+ such that $T_n \to +\infty$. To each n associate the neighborhood $S(x\ ,1/n)$ of x. Then for each t_n there exists a $t_n > T_n$ such that $xt_n \in S(x,1/n)$. Clearly $t_n \to +\infty$ and $xt_n \to x$. Thus $x \in \Lambda^+(x)$. Conversely, let U be a neighborhood of x. Since $x \in \Lambda^+(x)$ there is a sequence $\{t_n\}$ in R^+ with $t_n \to +\infty$ such that $xt_n \to x$. Thus xt_n must eventually be in every neighborhood of x so that, in particular, given any $T > 0$ there exists a $t_n > T$ such that $xt_n \in U$. Next, we establish the equivalence of statements

3 and 4. Let $x \in \Lambda^+(x)$. Then since $\Lambda^+(x)$ is positively invariant, we have $\gamma^+(x) \subset \Lambda^+(x)$. Since $\mathrm{cl}\ (\gamma^+(x)) = \gamma^+(x) \cup \Lambda^+(x)$, it follows that $\mathrm{cl}\ (\gamma^+(x)) = \Lambda^+(x)$. Conversely, let $\mathrm{cl}\ (\gamma^+(x)) = \Lambda^+(x)$. But since we know that $\mathrm{cl}\ (\gamma^+(x)) = \gamma^+(x) \cup \Lambda^+(x)$, it follows that $\gamma^+(x) \subset \Lambda^+(x)$ and hence that $x \in \Lambda^+(x)$. To see the equivalence of statements 3 and 5, let $x \in \Lambda^+(x)$. Then, since $\Lambda^+(x)$ is invariant, $xt \in \Lambda^+(x)$ for all $t \in R$. Thus $\gamma(x) \subset \Lambda^+(x)$. Conversely, let $\gamma(x) \subset \Lambda^+(x)$. Then for any $t \in R$, $xt \in \Lambda^+(x)$. In particular, for $t = 0$ we have $x \in \Lambda^+(x)$. Finally, we prove the equivalence of statements 3 and 6. To this end let $x \in \Lambda^+(x)$ and let $\varepsilon > 0$. Then there exists a sequence $\{t_n\}$ in R^+ with $t_n \to \infty$ such that $xt_n \to x$. Without loss of generality, we may assume $t_n \geq 1$ for all n. Then since $xt_n \to x$, this sequence must eventually be in every neighborhood of x. Hence there exists a $t_n \geq 1$ such that $xt_n \in S(x,\varepsilon)$. Conversely, choose a positive null sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$. Then for each n, we have a $t_n \geq 1$ such that $xt_n \in S(x,\varepsilon_n)$. Since $\varepsilon_n \to 0$, we have that $xt_n \to x$. Consider the sequence $\{t_n\}$ in R^+. Either $t_n \to +\infty$ or it contains a convergent subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to t \in R^+$. If $t_n \to +\infty$, $x \in \Lambda^+(x)$ by definition. If $t_{n_k} \to t$, then $xt_{n_k} \to xt$. But since $xt_n \to x$, we have $x = xt$ and therefore x is periodic with period t. It follows that $x \in \Lambda^+(x)$ and the proof is complete. //

From this theorem it should be clear that whenever a point x is positively Poisson stable then so is xt for every $t \in R$. This is true because if x is positively Poisson stable, then $x \in \Lambda^+(x)$ which implies $xt \in \Lambda^+(x)$ for all $t \in R$ by the invariance of $\Lambda^+(x)$. But $\Lambda^+(x) = \Lambda^+(xt)$ for all $t \in R$ by Theorem 3.5. Hence $xt \in \Lambda^+(xt)$ and xt is positively Poisson stable.

According to the above theorem, a point x is positively Poisson stable if $x \in \Lambda^+(x)$. Similarly, a point x is defined to be negatively Poisson stable if $x \in \Lambda^-(x)$. It is said to be Poisson stable if it is both positively and negatively Poisson stable. If a point x is Poisson stable, then both its motion and its trajectory are said to be Poisson stable.

From the preceding theorem observe that if $\mathrm{cl}\ (\gamma^+(x)) = \Lambda^+(x)$, then x is positively Poisson stable. Now consider the situation where $\gamma^+(x) = \Lambda^+(x)$. In that case the point x is positively Poisson stable since $\mathrm{cl}\ (\gamma^+(x)) = \gamma^+(x) \cup \Lambda^+(x) = \Lambda^+(x)$. We will now establish that $\gamma^+(x) = \Lambda^+(x)$ if and only if x is a periodic point. For if x is a periodic point, it is clear that $\gamma^+(x) = \Lambda^+(x)$. On the other hand, if $\gamma^+(x) = \Lambda^+(x)$, then $x \in \Lambda^+(x)$ and by the invariance of $\Lambda^+(x)$, $\gamma(x) = \Lambda^+(x)$. Then $xt' \in \gamma^+(x)$ for $t' < 0$ and therefore there exists a $T \geq 0$ such that $xt' = xT$. Hence $xt'(t) = xt(t') = xT(t)$.

Therefore $xt = x(t + T - t')$ for all $t \in R$, showing that x is periodic with period $T - t' > 0$.

With this last fact it is a natural question to ask whether there are Poisson stable points which are not periodic. The following example shows that such points do exist.

EXAMPLE 4.3 Consider the dynamical system defined on a torus by the system

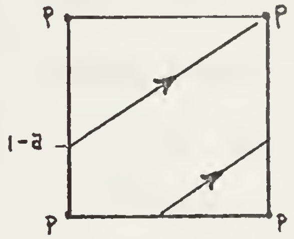
$$\frac{dx}{dt} = f(x,y),$$

$$\frac{dy}{dt} = af(x,y),$$

where $f(x,y)$ is periodic in both x and y with period 1. That is $f(x,y) = f(x + 1,y + 1) = f(x + 1,y) = f(x,y + 1)$. Further, we have that $f(x,y) > 0$ for x and y not both 0 mod 1 and $f(0,0) = 0$. The constant a is an irrational number.

Clearly the point $p = \binom{0}{0}$ is a critical point because at this point $\frac{dx}{dt} = 0 = \frac{dy}{dt}$. Solving this system simultaneously we have $\frac{dx}{dy} = 1/a$ which implies $y = ax + c$ where c is some constant of integration. Thus the trajectories describe straight lines of slope a in the plane. We may assume $a > 0$. Now no trajectory can contain p except $\gamma(p) = \{p\}$ since p is a rest point and since $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are both positive for all other points.

We note now that the function $y = ax + c$ is not defined at the point p since at that point $\frac{dx}{dy} = \frac{0}{0}$. So we claim that there is only one trajectory which approaches the point p as $t \to \infty$. The following diagram will help us to determine this trajectory. For purposes of this diagram, we assume $0 < a < 1$.



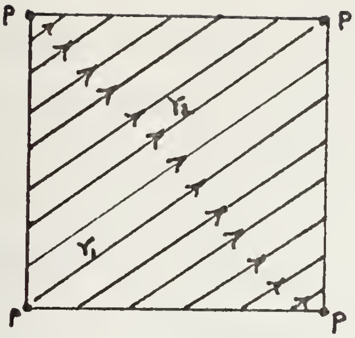
Consider the trajectory $y = ax + (1 - a)$. As $x \to 1$, it is clear that $y \to 1$. Hence this trajectory approaches the point $\binom{1}{1}$ as $t \to \infty$. But on the torus, the point $p = \binom{0}{0} = \binom{1}{1}$. Hence this trajectory approaches p as $t \to \infty$. Generalizing this approach to the case of an arbitrary positive value for a, the trajectory which approaches p as $t \to \infty$ is given by the equation $y = ax + (1 - (a \bmod 1))$. To verify this fact, let $x \to 1$. Then $y \to a + (1 - (a \bmod 1)) = 1 + (a - (a \bmod 1))$. But $a - (a \bmod 1)$ is just some integer n and hence as $x \to 1$, we have $y \to n + 1$. Thus as $t \to \infty$, this trajectory approaches the point $\binom{n+1}{1}$ but again, in the torus, this point and the point p are identified. Thus this is precisely the trajectory which approaches p as $t \to \infty$. Call this trajectory γ_2. Similarly there is only one trajectory which approaches the point p as $t \to -\infty$ It is given precisely by $y = ax + 0$. It is clear

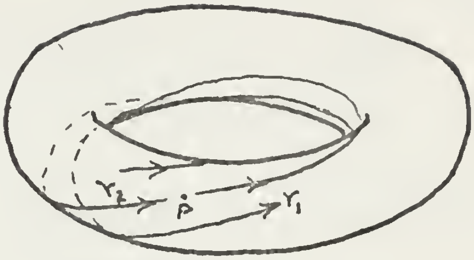
that as $x \to 0$, we have $y \to 0$ and this verifies that this trajectory does in fact approach p as $t \to -\infty$. Call this trajectory γ_1.

Now we claim that no trajectory in the torus is periodic. If there is such a trajectory, then it will pass through the same point twice and without loss of generality, we may assume that this point lies on the y-axis. We may make this assumption because the trajectory must intersect the y-axis. Assume that this point is the point $\binom{0}{c}$ where $0 < c < 1$. Then if the trajectory given by $y = ax + c$ intersects this point twice, we have $\binom{0}{c} = \binom{n}{an + c} \bmod 1$ for some $n > 0$. But this implies that $c = (an + c) \bmod 1$. Since $0 < c < 1$, we know that $c = c \bmod 1 = (m + c) \bmod 1$ where m is any integer. Hence it follows that $(an + c) \bmod 1 = (m + c) \bmod 1$. This implies that $an = k$ for some integer k. Now since $n \neq 0$, it follows that $a = k/n$. But this implies that a is a rational number which is a contradiction. Thus no trajectory can pass through the same point twice and hence no point except p is periodic.

We also note here that the trajectories γ_1 and γ_2 are not the same for if they were, then the equations $y = ax + (1 - (a \bmod 1))$ and $y = ax + 0$ must be equal. But this implies that $a \bmod 1 = 1$ and thus a must be an integer which contradicts the fact that a is irrational.

We now sketch several of the trajectories in the following diagram.





To complete our analysis of this example, we see that $\Lambda^+(\gamma_2) = \{p\}$ and that $\Lambda^-(\gamma_1) = \{p\}$. For the point p, we have $\Lambda^+(p) = \Lambda^-(p) = \{p\}$. For any other trajectory, we have $\Lambda^+(\gamma) = \Lambda^-(\gamma) =$ the torus since these trajectories are not periodic. Also, $\Lambda^+(\gamma_1) = \Lambda^-(\gamma_2) =$ the torus. Hence we have that all the points on γ_1 are positively Poisson stable since for each $x \in \gamma_1$, $x \in \Lambda^+(x)$. Also points on γ_2 are negatively Poisson stable. All other points are Poisson stable.

If we change this example so that $f(x,y) > 0$ for all x and y, then we will have no rest points and no periodic trajectories. Hence every trajectory is dense in the torus and the positive and negative limit sets of each point is the torus.

We now introduce the notion of a non-wandering point.

DEFINITION 4.4 A point $x \in X$ is said to be non-wandering if every neighborhood U of x is self positively recursive.

It is clear from this definition that if a point is positively Poisson stable then it is non-wandering. This idea can also be seen in the following theorem which gives some characterizations of non-wandering points.

THEOREM 4.5 *For any $x \varepsilon X$, the following are equivalent.*

1. *The point x is non-wandering,*
2. *The point x is an element of its positive prolongational limit set $J^+(x)$,*
3. *Every neighborhood of x is self negatively recursive,*
4. *The point x is an element of its negative prolongational limit set $J^-(x)$.*

PROOF Assume x is non-wandering. Consider a null sequence $\{\varepsilon_n\}$, $0 < \varepsilon_n$, $\varepsilon_n \to 0$, and a sequence $\{t_n\}$ in R with $t_n \to +\infty$. Since each $S(x,\varepsilon_n)$ is self positively recursive, we have an $x_n \varepsilon S(x,\varepsilon_n)$ and a $\tau_n > t_n$ with $x_n\tau_n \varepsilon S(x,\varepsilon_n)$. Since $\varepsilon_n \to 0$ we have $x_n \to x$ and $x_n\tau_n \to x$ and since $\tau_n \to +\infty$ we conclude that $x \varepsilon J^+(x)$. Thus statement 2 holds. Assume $x \varepsilon J^+(x)$. Then there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R with $x_n \to x$ and $t_n \to +\infty$ such that $x_nt_n \to x$. Now for any neighborhood U of x and $T > 0$ there is an N such that $t_n > T$, $x_n \varepsilon U$, and $x_nt_n \varepsilon U$ for $n \geq N$. Thus U is self positively recursive and, consequently, x is non-wandering. The equivalence of statements 3 and 4 is proved in the same manner. It remains to show that

statements 2 and 4 are equivalent. This follows directly from Theorem 3.17. //

We shall now prove some theorems which establish the connection between Poisson stable points and non-wandering points.

THEOREM 4.6 Let $x \in X$. Every $y \in \Lambda^+(x)$ is non-wandering.

PROOF It is necessary to show that if $y \in \Lambda^+(x)$ for some $x \in X$, then $y \in J^+(y)$. If $y \in \Lambda^+(x)$, then there is a sequence $\{t_n\}$ with $t_n \to +\infty$ and $xt_n \to y$. Since $t_n \to +\infty$, we may assume, if necessary by taking a subsequence, that $t_{n+1} - t_n \geq n$ for each n. Then setting $\tau_n = t_{n+1} - t_n$ and $xt_n = x_n$ we have $x_n \to y$, $x_n(t_{n+1} - t_n) = xt_{n+1} \to y$, and $\tau_n = t_{n+1} - t_n \to +\infty$. Thus $y \in J^+(y)$ and y is non-wandering. //

THEOREM 4.7 Let $P \subset X$ be such that every $x \in P$ is either positively or negatively Poisson stable. Then every $x \in \text{cl } P$ is non-wandering.

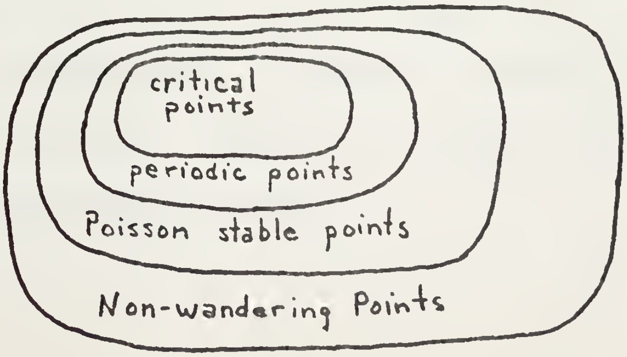
PROOF Let $\{x_n\}$ in P and $x_n \to x$. We must prove that $x \in J^+(x)$. For each n we have that either $x_n \in \Lambda^+(x_n)$ or $x_n \in \Lambda^-(x_n)$. Thus by taking a subsequence if necessary, we may assume that $x_n \in \Lambda^+(x_n)$ for all n or that $x_n \in \Lambda^-(x_n)$ for all n. In the first case, for each n there is a $t_n > n$ with $d(x_n, x_n t_n) < 1/n$. Then clearly $d(x, x_n t_n) \leq d(x, x_n) + d(x_n, x_n t_n) \leq d(x, x_n) + 1/n$. This

shows that $x_n t_n \to x$ and consequently $x \in J^+(x)$. In the second case similar considerations show that $x \in J^-(x)$. Thus by Theorem 4.5, every $x \in$ cl P is non-wandering. //

A partial converse of this last theorem will now be presented without proof. The proof is entirely constructive but not particularly instructive. A proof may be found in Bhatia and Szego.

THEOREM 4.8 Let X be complete and let every $x \in X$ be non-wandering. Then the set of Poisson stable points P is dense in X.

To see how several of these concepts are related, we present the following summary. It is clear that a critical point is a periodic point. Also, all periodic points are Poisson stable since the positive and negative limit sets of a periodic point are the periodic trajectory. Certainly the periodic point is contained in its periodic trajectory. Finally, all Poisson stable points are non-wandering points points for if x is a Poisson stable point, then $x \in \Lambda^+(x) \subset J^+(x)$. These ideas are summarized in the following diagram.



We now define a different type of stability. This notion will become important in discussing "dispersive" concepts.

DEFINITION 4.9 For any $x \in X$, the motion π_x is said to be positively Lagrange stable if cl $(\gamma^+(x))$ is compact. Further, if cl $(\gamma^-(x))$ is compact, the motion π_x is called negatively Lagrange stable. It is said to be Lagrange stable if cl $(\gamma(x))$ is compact.

If $X = R^n$, then the above statements are equivalent to the condition of the sets $\gamma^+(x)$, $\gamma^-(x)$, $\gamma(x)$ being bounded, respectively. The following statements about this notion are easily proved using Theorems 3.8 and 3.9 and Remark 3.10.

1. If X is locally compact, then a motion π_x is positively Lagrange stable if and only if $\Lambda^+(x)$ is a non-empty compact set.
2. If a motion π_x is positively Lagrange stable, then $\Lambda^+(x)$ is compact and connected.
3. If a motion π_x is positively Lagrange stable, then $d(xt,\Lambda^+(x)) \to 0$ as $t \to +\infty$.

We are now ready to discuss dynamical systems which are noted by their lack of the properties of recursiveness. One can easily draw a parallel between these dispersive concepts and the recursive concepts just defined. First, we need some definitions.

DEFINITION 4.10 Let $x \in X$.

1. The motion π_x is said to be positively Lagrange unstable whenever $\mathrm{cl}\,(\gamma^+(x))$ is not compact.
2. The motion is said to be negatively Lagrange unstable whenever $\mathrm{cl}(\gamma^-(x))$ is not compact.
3. The motion is said to be Lagrange unstable if it is both positively and negatively Lagrange unstable.
4. The point x is called positively Poisson unstable whenever $x \notin \Lambda^+(x)$.
5. The point x is called negatively Poisson unstable whenever $x \notin \Lambda^-(x)$.
6. The point x is called Poisson unstable whenever it is both positively and negatively Poisson unstable.
7. The point x is called wandering whenever $x \notin J^+(x)$.

This is a rather lengthy definition but the definitions are what we would expect. We are now ready to define dispersive concepts on the whole space X.

DEFINITION 4.11 The dynamical system (X,R,π) is said to be

1. Lagrange unstable if for each $x \in X$, the motion π_x is Lagrange unstable,
2. Poisson unstable if each $x \in X$ is Poisson unstable,
3. completely unstable if every $x \in X$ is wandering,

4. dispersive if for every pair of points $x,y \varepsilon X$, not necessarily distinct, there exist neighborhoods U_x of x and U_y of y such that U_x is not positively recursive with respect to U_y and U_y is not positively recursive with respect to U_x.

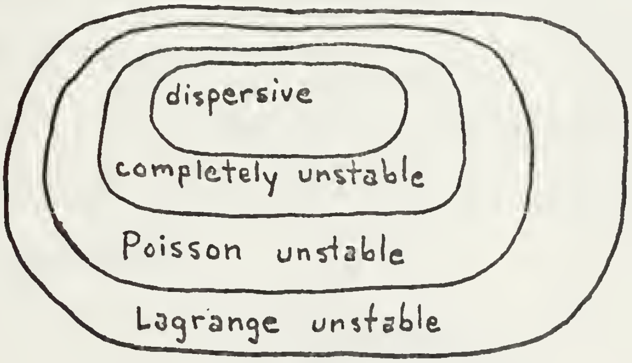
One may wonder why such concepts as positively and negatively wandering points are not defined. This is unnecessary since $x \varepsilon J^+(x)$ if and only if $x \varepsilon J^-(x)$. One can see from the definition that a completely unstable system must be Poisson unstable since if $x \not\varepsilon J^+(x)$, then $x \not\varepsilon \Lambda^+(x)$. The following remark proves the relationship between all the sets defined.

REMARK 4.12 In definition 4.11, each statement implies the preceding statement.

PROOF Suppose (X,R,π) is dispersive. Then for every pair of points $x,y \varepsilon X$, there exist neighborhoods U_x of x and U_y of y such that U_x is not positively recursive with respect to U_y. Set $x = y$. Then there exists a neighborhood of x which is not self positively recursive. Hence x is wandering so $x \not\varepsilon J^+(x)$. Since this is true for all $x \varepsilon X$, we have that X is completely unstable. Next suppose X is completely unstable. Then $x \not\varepsilon J^+(x)$ for all $x \varepsilon X$. Then also $x \not\varepsilon J^-(x)$ for all $x \varepsilon X$. Hence $x \not\varepsilon \Lambda^+(x)$ and $x \not\varepsilon \Lambda^-(x)$ since $\Lambda^+(x) \subset J^+(x)$ and $\Lambda^-(x) \subset J^-(x)$. Thus X is Poisson unstable. Finally assume X is Poisson unstable. Then

$x \not\in \Lambda^+(x)$ and $x \not\in \Lambda^-(x)$ for all $x \in X$. Suppose cl $(\gamma^+(x))$ is compact. Then by Theorem 3.8, $\Lambda^+(x)$ is non-empty and compact, hence complete. Then by Theorem 4.6 every point in $\Lambda^+(x)$ is non-wandering and by Theorem 4.8, the set P of Poisson stable points is dense in $\Lambda^+(x)$. Thus P is non-empty. But if q is a Poisson stable point in $\Lambda^+(x)$, then $q \in \Lambda^+(q)$ and the space is not Poisson unstable. Hence we have a contradiction so that cl $(\gamma^+(x))$ is not compact. A similar argument shows that cl $(\gamma^-(x))$ is not compact. Hence X is Lagrange unstable. //

The following diagram provides a summary of the ideas in the preceding remarks.

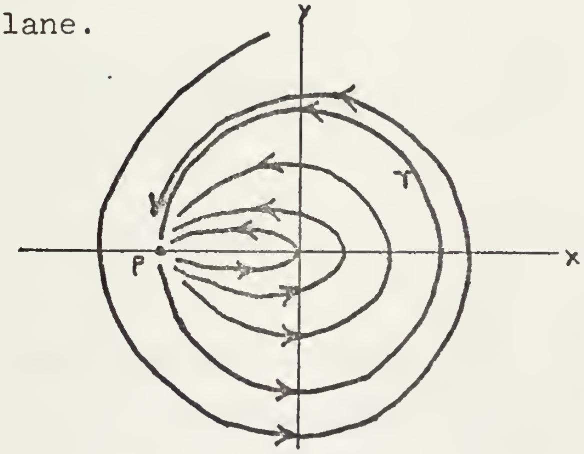


Thus we know that a dispersive system is completely unstable, a completely unstable system is Poisson unstable and a Poisson unstable system is Lagrange unstable. One may wonder if the reverse implications hold. In general they do not as the following examples illustrate.

To show that a Lagrange unstable system may not be Poisson unstable we want $x \in \Lambda^+(x)$ but cl $(\gamma^+(x))$ not

compact. Since $x \in \Lambda^+(x)$, we know that $\gamma^+(x) \subset \Lambda^+(x)$ and hence cl $(\gamma^+(x)) = \Lambda^+(x)$. Thus we must have an example where $\Lambda^+(x)$ is not compact and yet $x \in \Lambda^+(x)$. To this end, consider Example 4.3 of the dynamical system defined on the torus. We delete the point p from the torus so that the torus is no longer compact. Then for any x in the torus such that $x \notin \gamma_2$, we have $\Lambda^+(x)$ = the torus and hence $x \in \Lambda^+(x)$ but $\Lambda^+(x)$ is not compact. Thus cl $(\gamma^+(x))$ is not compact. Clearly for $x \in \gamma_1$, $\Lambda^-(x) = \emptyset$ and hence cl $(\gamma^-(x))$ is not compact. Similar results hold for $x \in \gamma_2$ and cl $(\gamma^+(x))$ is not compact. Thus for all x in the torus, cl $(\gamma^+(x))$ is not compact and cl $(\gamma^-(x))$ is not compact. Hence this system is Lagrange unstable but it is not Poisson unstable because for any $x \notin \gamma_2$, $x \in \Lambda^+(x)$.

To see that a Poisson unstable system may not be completely unstable, consider the following phase portrait in the x,y-plane.



The unit circle contains a rest point p and a trajectory γ such that for each point $q \in \gamma$ we have $\Lambda^+(q) = \Lambda^-(q) = \{p\}$.

All trajectories in the interior of the unit circle have the same positive and negative limit sets as γ. All trajectories in the exterior of the unit circle spiral to the unit circle as $t \to +\infty$, so that for each point q in the exterior of the unit circle we have $\Lambda^+(q) = \{p\} \cup \gamma$, and $\Lambda^-(q) = \emptyset$. If we consider the dynamical system obtained from this one by deleting the rest point p then the resultant system is Poisson unstable. However it is not completely unstable because for each $q \in \gamma$, $J^+(q) = \gamma$ so $q \in J^+(q)$.

Finally to see that a completely unstable system may not be dispersive consider the following dynamical system defined by the differential equations

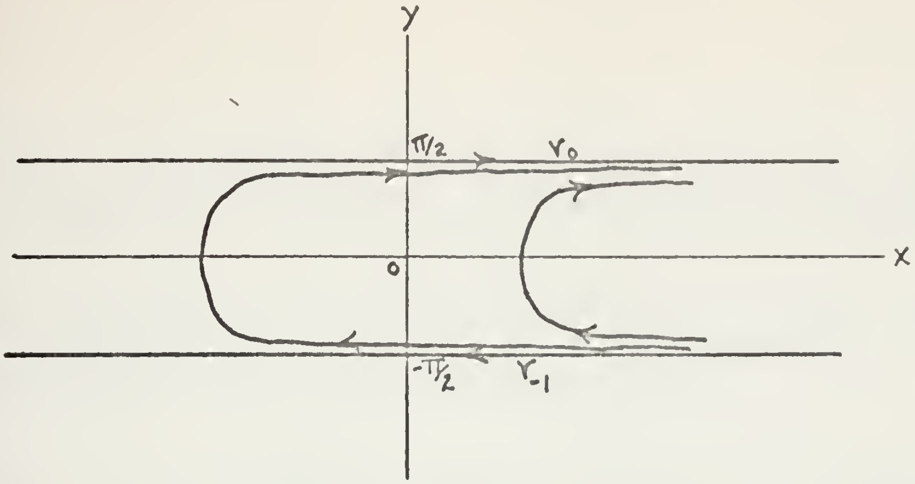
$$\dot{x} = \sin y$$

$$\dot{y} = \cos^2 y$$

This system contains trajectories γ_k given by

$$\gamma_k = \{(x,y): y = k\pi + \pi/2\}, \quad k = 0, \pm 1, \pm 2, \ldots$$

These are lines parallel to the x-axis. Between any two consecutive γ_k's the trajectories are given by $\gamma = \{(x,y): x + c = \sec y\}$, where c is a constant depending on the trajectory. The phase portrait between the lines $y = -\pi/2$ and $y = \pi/2$ is shown in the diagram below.



It can be seen that this system is completely unstable since for any $p \in \gamma_{-1}$, $J^+(p) = \gamma_0$ and for all other points p not belonging to any γ_k, $J^+(p) = \emptyset$. Thus $p \notin J(p)$ for any p and hence every point is wandering. However if $p \in \gamma_{-1}$ and $q \in \gamma_0$, then every neighborhood of q is recursive with respect to any neighborhood of p and the system is not dispersive.

The following result shows that to prove Lagrange instability, it is sufficient to prove that the space is positively Lagrange unstable.

THEOREM 4.13 A dynamical system (X,R,π) is Lagrange unstable if and only if it is positively Lagrange unstable.

PROOF The necessity is trivial. On the other hand, let the space be positively Lagrange unstable. Then for each $x \in X$, cl $(\gamma^+(x))$ is not compact. Suppose cl $(\gamma^-(x))$ is compact. Then $\Lambda^-(x)$ is non-empty and compact. Since $\Lambda^-(x)$ is invariant, $yt \in \Lambda^-(x)$ for all $t \in R^+$ and $y \in \Lambda^-(x)$. Then since $\Lambda^-(x)$ is closed, cl $(\gamma^+(y)) \subset \Lambda^-(x)$ and hence cl $(\gamma^+(y))$ is compact. This is a contradiction. Hence

cl $(\gamma^{-}(x))$ is not compact for each $x \varepsilon X$. Thus the space is also negatively Lagrange unstable, yielding the result. //

For a dynamical system defined in the x,y-plane, the following theorem provides an important fact.

THEOREM 4.14 For dynamical systems defined by a system of differential equations in the euclidean plane, the concepts of Lagrange instability and complete instability are equivalent.

PROOF Let this system be given by the differential equations

$$\dot{x} = f(x,y),$$

$$\dot{y} = g(x,y),$$

where $f(x,y)$ and $g(x,y)$ are continuous functions.

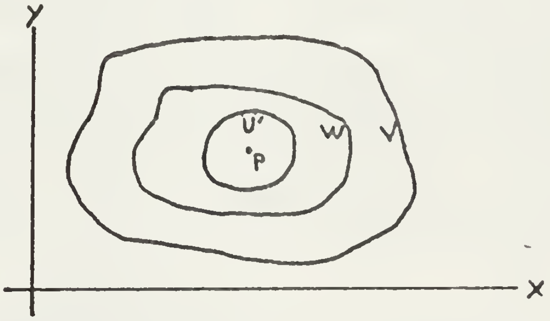
Clearly if this system is completely unstable, then it is Lagrange unstable. So assume the system is Lagrange unstable and that there is a point p which is non-wandering. Then $p \varepsilon J^{+}(p)$. The point p cannot be a critical point for then cl $(\gamma^{+}(p))$ would be compact contrary to hypothesis. Likewise, there exists a neighborhood V of non-critical points about the point p.

Let $p = \binom{p_1}{p_2}$. Since p is not a critical point, then either $f(p_1,p_2) \neq 0$ or $g(p_1,p_2) \neq 0$. Assume $f(p_1,p_2) \neq 0$.

Since $f(x,y)$ is continuous, there is a neighborhood W about p where $f(x,y) \neq 0$ for all points $\binom{x}{y} \in W$. Now consider the function $H:R^2 \to R$ defined by

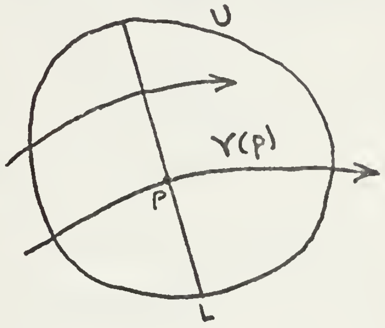
$H(x,y) = \tan^{-1}\left[\frac{g(x,y)}{f(x,y)}\right]$. Clearly $H(x,y)$ is continuous in W. Since $H(x,y) = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left(\frac{dy}{dx}\right)$, we may consider the function H to be a "slope" function since it provides us with the slope of any trajectory at a given point.

Now by the continuity of $H(x,y)$ there is a neighborhood U' such that for all points $\binom{x}{y}$ in U', $|H(x,y) = H(p_1,p_2)| < \varepsilon$, where $\varepsilon > 0$ is chosen such that $U' \subset W$. The relationship between the sets V, W, and U' is shown in the following diagram.

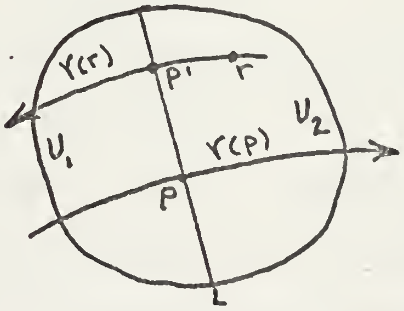


Now since $|H(x,y) - H(p_1,p_2)| < \varepsilon$ for all $\binom{x}{y}$ in U', we see that the neighborhood U' contains only points whose trajectories have slopes that differ from the slope of $\gamma(p)$ at p by less than ε. Now let $\delta > 0$ be so chosen such that $\delta < \varepsilon$ and $\delta < 1/10$. Now let $U = \{\binom{x}{y} \in U' : |H(x,y) - H(p_1,p_2)| < \delta\}$. The number 1/10 was chosen so that we are assured that we can construct the following "transversal" through all the trajectories lying within U.

Let L be a line segment through the point p making an angle of 90° with the motion through p. Such a line can be constructed since we are working in the plane. Now every trajectory in U must cross this line L by our above construction of U.

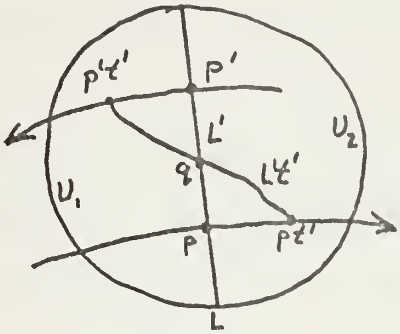


We claim that each trajectory crossing L in U must do so in the same direction as $\gamma(p)$. To see this, let $r \in U$ and suppose $\gamma(r)$ crosses L in the opposite direction of $\gamma(p)$. Let p' be the point where $\gamma(r)$ meets the line L. Now let $U-L = U_1 \cup U_2$. It is clear that U_1 and U_2 are disjoint sets.

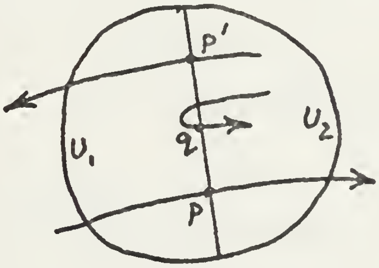


Since $\gamma(r)$ and $\gamma(p)$ cross L in different directions, we may assume that for some small time $t' > 0$, $p't' \in U_1$ and $pt' \in U_2$. Now let L' be the segment of L which joins the points p and p' and consider $\pi(L',t')$. Since L' is a connected set, $\pi(L',t')$ is a connected set. Also part

of L't' will be in U_1 and part of L't' will be in U_2 since p't' ε U_1 and pt' ε U_2.



Now L't' is a connected set so there is at least one point q such that q ε L'$\cap$ L't'. Thus it follows that q = qt' since it would be impossible that any other trajectory passing through L' at time t = 0 would cross the point q in time t = t'. For this to happen, we would have the following situation.

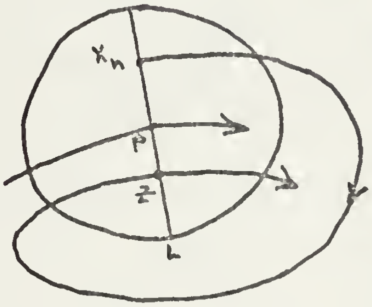


Clearly this is an impossibility in U and hence we have q = qt'. But this implies that q is a periodic point and hence cl $(\gamma^+(q))$ is compact and we have a contradiction. Hence all trajectories cross L in the same direction.

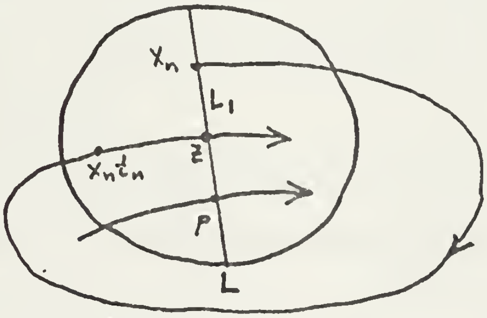
Now p ε $J^+(p)$ so there exist sequences $\{x_n\}$ in X and $\{t_n\}$ in R such that $x_n \to p$ and $t_n \to +\infty$ with $x_n t_n \to p$. If any of the x_n's are periodic, then cl $(\gamma^+(x_n))$ is compact and we have a contradiction. Hence no x_n is periodic and, without loss of generality, we may assume each x_n lies on

L. Now for n sufficiently large, we may assume that $x_n t_n \in U$ since $x_n t_n \to p$. Then it follows that there exists a $T \in R$ such that $x_n t_n(T)$ lies on L. Call this point z.

It is clear that z lies between the points x_n and p, for if not we have the situation shown below and it is clear that in this case, $x_n t_n \to p$ is an impossibility.

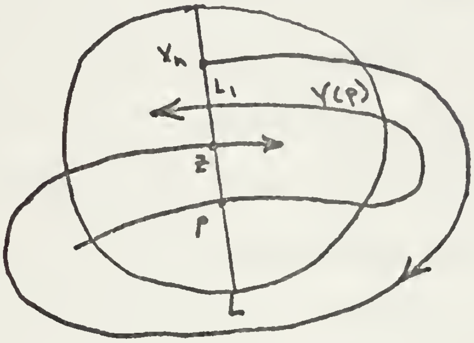


Thus we must have the following situation.



Now, let L_1 be the segment of L which joins the points x_n and z. Since $z = x_n(t_n + T)$, we consider the set $\{x_n[0,t_n + T] \cup L_1\} = A$. This is a Jordan curve and hence the interior of A together with A is a compact subset of R^2. Let B denote this compact set. Now $p \in B$ so we consider $\gamma^+(p)$. We claim that $pt \in B$ for all $t \in R^+$. If this is not the case, then $\gamma^+(p)$ must cross the set A at some point since $\gamma^+(p)$ is a connected set. But $\gamma^+(p)$

cannot cross A anywhere along $x_n[0,t_n + T]$ because different trajectories cannot intersect. Then the only alternative is that $\gamma^+(p)$ must cross L_1.



But this too is impossible since to do so would mean crossing L in the opposite direction from $\gamma(x_n)$. Hence $pt \in B$ for all $t \in R^+$. Hence $\gamma^+(p) \subset B$ and since B is compact, we have cl $(\gamma^+(p)) \subset B$. Therefore cl $(\gamma^+(p))$ is compact. Thus we have our final contradiction and the assumption that $p \in J^+(p)$ is untenable. Thus the system is completely unstable. //

(The above proof was communicated to me through a private conversation with Dr. William Chewning.)

We now continue with several characterizations of dispersive systems.

THEOREM 4.15 A dynamical system (X,R,π) is dispersive if and only if $J^+(x) = \emptyset$ for each $x \in X$.

PROOF Let the system be dispersive. Assume there is a point $x \in X$ such that $J^+(x) \neq \emptyset$. Then if $y \in J^+(x)$ there are sequences $\{x_n\}$ and $\{t_n\}$ such that $x_n \to x$,

$t_n \to +\infty$ and $x_n t_n \to y$. This shows that for any neighborhoods U_x of x and U_y of y, $U_x t_n \cap U_y \neq \emptyset$ as the element $x_n t_n$ is contained in this intersection. But this contradicts the assumption that the system is dispersive because U_y is then positively recursive with respect to U_x. Hence $J^+(x) = \emptyset$ for all $x \in X$. Conversely, let $J^+(x) = \emptyset$ for all $x \in X$. We claim that there exist neighborhoods U_x of x and U_y of y and a $T \geq 0$ such that $U_x t \cap U_y = \emptyset$ for all $t \geq T$. For if not there exist sequences $\{x_n\}$ $\{y_n\}$ $\{t_n\}$ such that $x_n \to x$, $y_n = x_n t_n$, $y_n \to y$, and $t_n \to +\infty$, so that $y \in J^+(x)$. But this is a contradiction. Similarly U_y is not positively recursive with respect to U_x. Hence the system is dispersive. //

THEOREM 4.16 For a given (X,R,π) the following are equivalent.

1. The space (X,R,π) is dispersive.
2. For any two points x,y in X there are neighborhoods U_x of x and U_y of y and a constant $T > 0$ such that $U_x \cap U_y t = \emptyset$ for all t, $|t| \geq T$.
3. For any two distinct points x,y in X, $y \notin J^+(x)$ and $x \notin J^+(y)$.

PROOF Let (X,R,π) be dispersive. Then for any two points x,y in X, there exist neighborhoods U_x and U_y such that U_x is not positively recursive with respect to U_y and U_y is not positively recursive with respect to U_x. Hence for some $T > 0$, $U_x \cap U_y t = \emptyset$ for all $t > T$ and $U_x t \cap U_y = \emptyset$ for all $t > T$ which implies $U_x \cap U_y(-t) = \emptyset$. Thus $U_x \cap U_y t = \emptyset$ for all t, $|t| > T$.

Assume statement 2 and suppose $x \in J^+(y)$. Then there exist sequences $\{y_n\}$ and $\{t_n\}$ such that $y_n \to y$ and $t_n \to +\infty$ and $y_n t_n \to x$. For n sufficiently large we may assume that $y_n \in U_y$ and $y_n t_n \in U_x$ and $t_n > T$. Then we have $U_x \cap U_y t_n \neq \emptyset$ which is a contradiction. The proof that $y \not\in J^+(x)$ is similar.

Finally assume that for any two distinct points x,y in X, $y \not\in J^+(x)$ and $x \not\in J^+(y)$. Suppose for all neighborhoods U_x and U_y and any $T > 0$ we have $U_x \cap U_y t \neq \emptyset$ for some $t > T$. Since this is true for all neighborhoods, it will be true for a nested sequence of neighborhoods about the points x and y. Hence consider for each n, the neighborhoods $S(x,1/n)$ and $S(y,1/n)$. Now set $T = T_1$. Then $S(x,1) \cap S(y,1)t_1 \neq \emptyset$ for some $t_1 > T_1 + 1$. Set $T_2 = t_1$. Then there exists a $t_2 > T_2 + 2$ such that $S(x,1/2) \cap S(y,1/2)t_2 \neq \emptyset$. In this manner we have constructed sequences $\{x_n\}$, $\{y_n\}$, $\{t_n\}$, with $x_n \to x$, $t_n \to \infty$, $x_n t_n = y_n$, and $y_n \to y$. We note that the process in constructing the sequence $\{t_n\}$ cannot terminate for if it did, then we have found a $T_k > 0$ such that $S(x,1/k) \cap S(y,1/k)t = \emptyset$ for all $t > T_k$ contrary to our hypothesis. Thus we have $x_n t_n \to y$ so $y \in J^+(x)$ but this is a contradiction and hence U_x cannot be positively recursive with respect to U_y. A similar argument holds for U_y not positively recursive with respect to U_x. Hence (X,R,π) is dispersive. //

We now introduce the final concept in this section. It is that of parallelizable dynamical systems. We are headed toward one particular theorem and consequently many lemmas and theorems will be presented without proof since their proofs are constructive in nature or are not particularly informative. However, the proofs may be found in Bhatia and Szego pages 48 through 53. Formally we begin with the definition of a parallelizable dynamical system.

DEFINITION 4.17 A dynamical system (X,R,π) is called parallelizable if there exists a set $S \subset X$ and a homeomorphism $h: X \to S \times R$ such that $SR = X$ and $h(xt) = (x,t)$ for every $x \in S$ and $t \in R$.

To proceed in the study of parallelizable dynamical systems we need to develop a theory of sections.

DEFINITION 4.18 A set $S \subset X$ is called a section of (X,R,π) if for each $x \in X$ there is a unique time $\tau(x)$ such that $x\tau(x) \in S$.

The set S in Definition 4.17 is in fact a section of (X,R,π) for if z is any element in X, then $z = xt$ for some $x \in S$ and $t \in R$. If this was not the case, then $SR \neq X$. Then the value $(-t)$ is precisely $\tau(z)$ given in Definition 4.18. It is clear that $z\tau(z) = z(-t) = x \in S$. To see that $(-t)$ is unique, assume there exists $(-t') \in R$ such that $z(-t') = y \in S$. Then $xt = yt'$. Now using the homeomorphism

h, we have $h(xt) = h(yt')$ which implies $(x,t) = (y,t')$. Thus we have $x = y$ and $t = t'$. Thus $\tau(z)$ is unique.

Then to summarize, it is clear that if a dynamical system is parallelizable it must have a subset S which contains a point of every trajectory so that $SR = X$. Moreover, the set S can contain only one point of every trajectory for otherwise $\tau(x)$ would not be unique. To see that this last statement is true, suppose there is a trajectory which intersects the section twice. Then there are points x,y in this trajectory such that $x,y \in S$ and $xt = y$ for some $t > 0$. Then $\tau(x) = 0$ since $x0 = x \in S$ but also $xt = y \in S$ and hence $t = \tau(x)$. Since $t > 0$, this contradicts the uniqueness of $\tau(x)$. This implies in particular that the dynamical system cannot have any critical points or periodic trajectories. To see how a parallelizable system compares with the other dispersive concepts defined in this section, consider the following example.

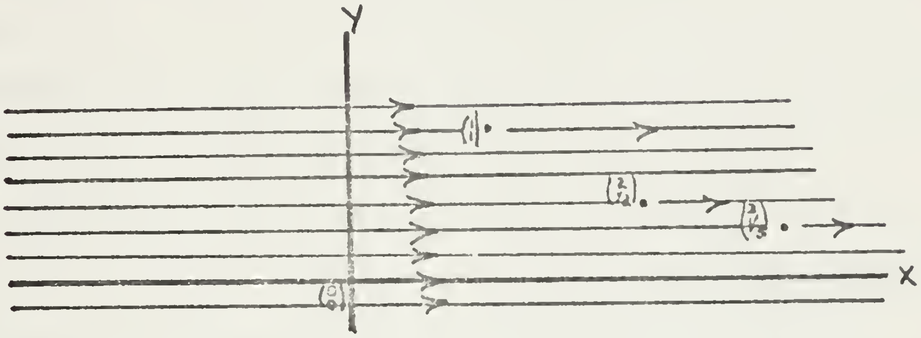
EXAMPLE 4.19 Consider a dynamical system defined in R^2 by the differential equations

$$\frac{dx}{dt} = f(x,y),$$

$$\frac{dy}{dt} = 0,$$

where $f(x,y)$ is continuous, and moreover $f(x,y) = 0$ whenever the point $\binom{x}{y}$ is of the form $\binom{n}{1/n}$ with n a positive integer.

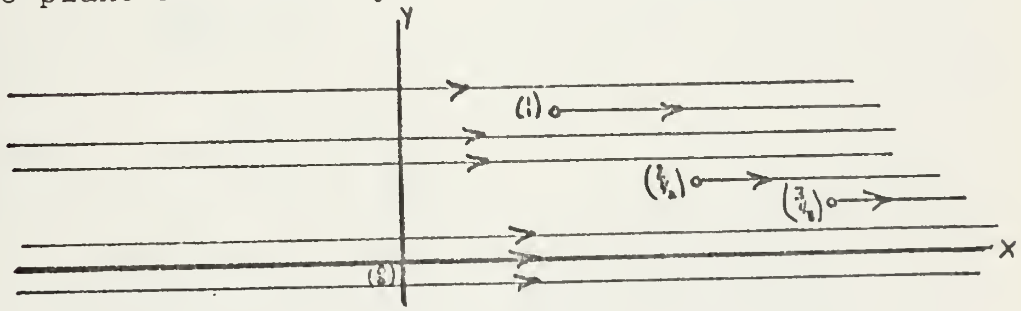
For simplicity we assume that $f(x,y) > 0$ for all other points. This system is sketched in the diagram below.



Now we form the system in which we are interested by deleting the sets

$$I_n = \{\binom{x}{y}: x \leq n,\ y = 1/n\}, \quad n = 1, 2, 3, \ldots$$

from the plane R^2. This system is sketched below.



It is clear that this system is dispersive since the only way two neighborhoods U_x of x and U_y of y could be positively recursive with respect to each other would be if the points x and y lie on the same trajectory. Then clearly we could find a $T > 0$ for which $U_x \cap U_y t = \emptyset$ or $U_y \cap U_x t = \emptyset$ for all $|t| > T$. Hence this system is dispersive but it is not parallelizable since no section S can exist for which $\tau(x)$ would be continuous. This fact can be seen when it is realized that the only section possible

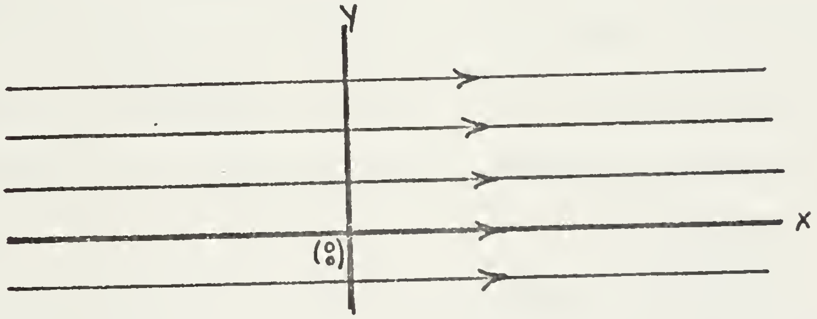
is one that looks like $y = 1/x + c$ for any constant c together with some section which intersects the trajectories below the x-axis. If this "lower" section intersects the trajectory on the x-axis, then $\tau(x)$ could not be continuous. If the lower section looks like $y = -1/x + c$, then $\tau(x)$ would be continuous but no section will intersect the trajectory on the x-axis. Hence the system is not parallelizable.

We can get an example of a parallelizable dynamical system from the previous example by considering the differential equations

$$\frac{dx}{dt} = 1,$$

$$\frac{dy}{dt} = 0.$$

This system is sketched below.



Clearly any line parallel to the y-axis is a section of the system.

These examples also give an intuitive feeling for the notion of a parallelizable dynamical system in that all trajectories are in some sense parallel.

The function $\tau(x)$ will be basic in our development. In general the function $\tau(x)$ need not be continuous however for a dynamical system to be parallelizable there must exist a section where $\tau(x)$ is continuous on X. This fact is proved in the following theorem.

THEOREM 4.20 A dynamical system (X,R,π) is parallelizable if and only if it has a section S with $\tau(x)$ continuous on X.

PROOF Suppose (X,R,π) has a section S with $\tau(x)$ continuous on X. Then clearly $SR = X$. We now define the homeomorphism $h: X \to S\times R$ by $h(x) = (x\tau(x), -\tau(x))$. To see that h is one-to-one let $h(x) = h(y)$ for x and y in X. Then $x\tau(x) = y\tau(y)$ and $-\tau(x) = -\tau(y)$. Hence $x\tau(x) = y\tau(x) \in S$ which implies that $x = y$ for otherwise $\tau(y)$ is not unique. Hence h is one-to-one. The function h is continuous by the continuity of $\tau(x)$ and the continuity axiom. The inverse $h^{-1}: S\times R \to X$ is given by $h^{-1}(x,t) = xt$ and is clearly one-to-one and continuous. Thus h is a homeomorphism so the definition of a parallelizable dynamical system is satisfied.

Now assume (X,R,π) is parallelizable. Then the set S in the definition is a section of X. Since for any $x \in X$, $x = yt$ for some $y \in S$ and $t \in R$, we set $\tau(x) = -t$. Then $x\tau(x) = x(-t) = y \in S$. The continuity of $\tau(x)$ follows from the continuity of h in the definition. //

To continue this development we need to define special types of sections. This is done in the following definition.

DEFINITION 4.21 An open set U in X will be called a tube if there exists a $\tau > 0$ and a subset $S \subset U$ such that $SI_\tau \subset U$ and for each $x \in U$ there is a unique $\tau(x)$, $|\tau(x)| < \tau$, such that $x\tau(x) \in S$. Here $I_\tau \equiv (-\tau, \tau)$. U is also called a τ-tube with section S, and S a $(\tau - U)$-section of the tube U. If $I_\tau = R$ then U is an ∞-tube and S an $(\infty - U)$-section.

DEFINITION 4.22 Given an open ∞-tube U with a section S and $\tau(x)$ continuous on U, and given sets N and K such that $N \subset K \subset S$ where N is open in S and K is compact, we shall call KR the compactly based tube over K. Then indeed $\tau(x)$ restricted to KR is continuous on KR.

Proceeding toward our final result, the next theorems and lemma are presented without proof.

THEOREM If X is locally compact and separable, and if every $x \in X$ is a wandering point, then there exists a countable covering $\{K_nR\}$ of X by compactly based tubes K_nR each with $\tau_n(x)$ continuous on K_nR.

LEMMA A compactly based ∞-tube U with section K of a dispersive dynamical system (X,R,π) is closed in X.

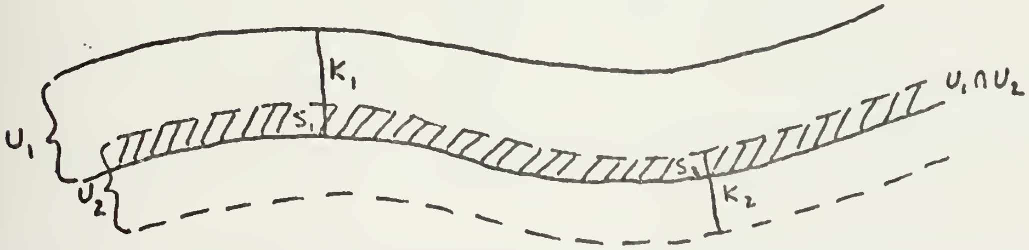
We also need the following important topological theorem.

TIETZE EXTENSION THEOREM *Let* X *be a normal topological space, let* A *be a closed subset of* X *and let* f *be a continuous function on* A *to the closed interval* $[a,b]$. *Then* f *has a continuous extension* g *which carries* X *into* $[a,b]$.

The final lemma needed to prove our result is quite instructive and is completely presented below.

LEMMA *Let* U_1, U_2 *be two compactly based tubes of a dispersive dynamical system with sections* K_1, K_2 *and continuous functions* $\tau_1(x)$ *and* $\tau_2(x)$ *respectively.* *If* $U_1 \cap U_2 \neq \emptyset$ *then* $U = U_1 \cup U_2$ *is a compactly based tube with a section* $K \supset K_1$ *and a continuous function* $\tau(x)$. *Moreover, if the time distance between* K_1 *and* K_2 *along trajectories in* $U_1 \cap U_2$ *is less than* τ (> 0), *the time distance between* K *and* K_2 *along trajectories in* U *is also less than* τ.

PROOF In this proof, the following diagram will be useful.



By the previous lemma, U_1 and U_2 are closed. They are invariant because each tube contains a section K_1 and K_2 respectively and $K_1R \subset U_1$ and $K_2R \subset U_2$. Hence $U_1 \cap U_2$ is invariant and closed. Further, $K_2 \cap U_1$ is compact and

non-empty, as shown in the diagram. Set $S_2 = K_2 \cap U_1$ and $S_1 = K_1 \cap U_2$. Any trajectory in $U_1 \cap U_2$ intersects S_1 in exactly one point and intersects S_2 in exactly one point. Thus for any $x \in U_1 \cap U_2$, $\tau_1(x) = \tau_2(x) + \tau_1(x\tau_2(x))$. This is true because $x\tau_1(x) = x\tau_2(x)(\tau_1(x\tau_2(x))) = x(\tau_2(x) + \tau_1(x\tau_2(x)))$ and there are no rest points or periodic trajectories in a dispersive system. The function τ_1 is continuous on S_2 which is compact and now we apply the Tietze extension theorem. This is possible since every metric space is normal. Thus the continuous function τ_1 is extended to a continuous function τ defined on K_2 where $\tau(x) = \tau_1(x)$ for $x \in S_2$. If $\tau_1(x) \in (-\tau,\tau)$ for $x \in S_2$, we have $\tau(x) \in (-\tau,\tau)$ for $x \in K_2$.

Notice now that $\{x\tau(x): x \in S_2\} = S_1$ and $\tau(x)$ being continuous, $\{x\tau(x): x \in K_2\}$ is compact as K_2 is compact. Set $K = K_1 \cup \{x\tau(x): x \in K_2\}$ and define $\tau^*(x)$ on $KR = K_1R \cup K_2R$ as follows:

$$\tau^*(x) = \begin{cases} \tau_1(x) \text{ for } x \in K_1R \\ \\ \tau_2(x) + \tau(x\tau_2(x)) \quad \text{for } x \in K_2R \end{cases}$$

$\tau^*(x)$ is continuous on KR and we need only verify that if $x \in U_1 \cap U_2$, then $\tau_1(x) = \tau_2(x) + \tau(x\tau_2(x))$ which has already been proved since $\tau(x) = \tau_1(x)$ for $x \in S_2$. Thus the lemma is proved. //

Finally, we prove the theorem that is the main focus of this section on parallelizable dynamical systems.

THEOREM 4.23 *A dynamical system* (X,R,π) *on a locally compact separable metric space* X *is parallelizable if and only if it is dispersive.*

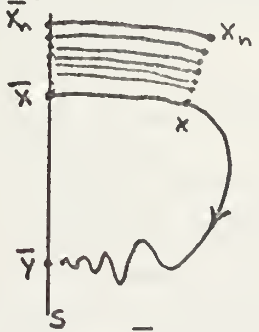
PROOF Let the dynamical system (X,R,π) be dispersive. By Theorem 4.20 it is sufficient to prove that X has a section S with $\tau(x)$ continuous on X. By a previous theorem, there is a countable covering $\{U_n\}$ of X by compactly based tubes U_n with sections K_n and continuous functions $\tau_n(x)$. We replace this covering by a like covering $\{U^n\}$ of compactly based tubes which we construct as follows. Set $K_1 = K^1$, and $U_1 = U^1$. Beginning with U^1 and U_2 we use the previous lemma to enlarge K^1 to a compact set K^2, thus obtaining the compactly based tube $U^2 = U^1 \cup U_2$ with $\tau^2(x)$ continuous on U^2. This leaves K_1 unaltered. Having found U^n in the same manner, we take it together with U_{n+1} and construct U^{n+1} with $K^{n+1} \supset K^n$, and $\tau^{n+1}(x)$ continuous on U^{n+1}. Now set $S = \bigcup K^n$, then $X = SR$ and the function $\tau(x)$ defined by $\tau(x) = \tau^n(x)$ for $x \in U^n$ is continuous on X, with the property that $x\tau(x) \in S$. Moreover $\tau(x)$ is unique for each $x \in X$. Thus X has a section S with continuous $\tau(x)$ defined on X. The system (X,R,π) is thus parallelizable.

Conversely, suppose the system (X,R,π) is parallelizable. Then there exists a section S with $\tau(x)$ continuous on X.

Suppose $J^+(x) \neq \emptyset$ for some $x \in X$. Let $y \in J^+(x)$. Then since $y \in X$, there exists $\overline{y} \in S$ and $\overline{t}_y \in R$ such that $\overline{yt}_y = y$. Now since $J^+(x)$ is invariant, $y(-\overline{t}_y) \in J^+(x)$ which implies that $\overline{y} \in J^+(x)$. Then there exist sequences $\{x_n\}$ and $\{t_n\}$ with $x_n \to x$, $t_n \to +\infty$, and $x_n t_n \to \overline{y}$. Now for each x_n, there exists $\overline{x}_n \in S$ and $\overline{t}_n \in R$ such that $\overline{x}_n \overline{t}_n = x_n$. Similarly there exists $\overline{x} \in S$ and $\overline{t}_x \in R$ such that $\overline{xt}_x = x$. Then since $x_n t_n \to \overline{y}$, it follows that $\overline{x}_n \overline{t}_n(t_n) \to \overline{y}$.

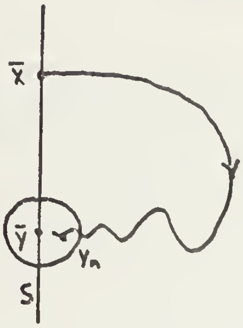
Consider for each fixed k, $k = 1, 2, \ldots$, the sequence $\{\overline{x}_n \overline{t}_n(t_k)\}$. Since $\overline{x}_n \overline{t}_n \to \overline{xt}_x$, we have by the continuity axiom that $\overline{x}_n \overline{t}_n(t_k) \to \overline{xt}_x(t_k)$. Hence we may assume by taking subsequences if necessary that $d(\overline{x}_n \overline{t}_n(t_k), \overline{xt}_x(t_k)) \leq 1/k$ for $k = 1, 2, \ldots$ Then it follows that $\overline{xt}_x(t_n) \to \overline{y}$ since $d(\overline{y}, \overline{xt}_x(t_n)) \leq d(\overline{y}, \overline{x}_n \overline{t}_n(t_n)) + d(\overline{x}_n \overline{t}_n(t_n), \overline{xt}_x(t_n)) \leq$

$d(\overline{y}, \overline{x}_n \overline{t}_n(t_n)) + 1/n$. Now since $\overline{xt}_x(t_n) \to \overline{y}$, let $\tau_n = t_x + t_n$. Since $t_n \to +\infty$, it follows that $\tau_n \to +\infty$. Hence we have that $\overline{y} \in \Lambda^+(\overline{x})$.



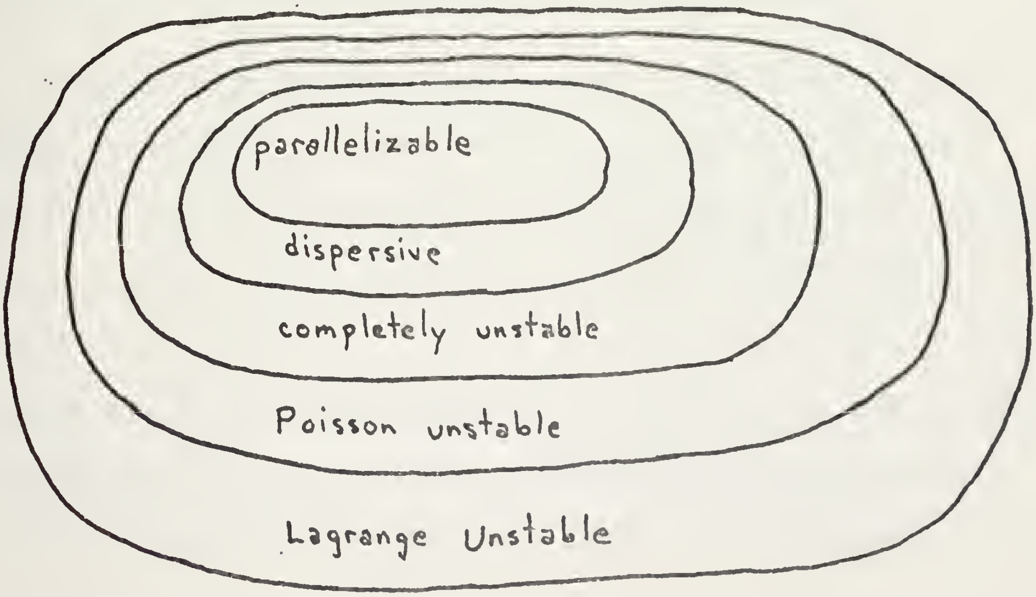
Now for each n set $y_n = \overline{x}\tau_n$. Then $y_n \to \overline{y}$. Since $\tau_n \to +\infty$, we may assume that $\tau_n > n$ for each n. Let U_y be an

arbitrarily small neighborhood of y. Then since $y_n \to \bar{y}$, $y_n \in U_y$ for all $n > N$ where N is a large enough integer.



We know that $\tau(x)$ is continuous and since $\tau(\bar{y}) = 0$, it follows that $\tau(y_n)$ is close to 0 for all $n > N$. Hence we may assume that $|\tau(y_n)| < 1/n$ for each $n > N$. But this is impossible since $\tau(y_n)$ is unique and we have already determined that $\tau(y_n) = -\tau_n < -n$ for all n. Thus we have a contradiction so it is impossible that $y \in J^+(x)$ and hence $J^+(x) = \emptyset$ for all $x \in X$. Then by Theorem 4.15 we have that (X,R,π) is dispersive. //

The relationships between the various dispersive concepts is now conveniently summarized below.



V. STABILITY THEORY

In this section we will study stability theory first from the abstract point of view that we have established and then from an applied point of view that uses systems of differential equations and studies the eigenvalues of such a system. To begin we need some basic definitions. Throughout this section, it will be assumed that the space X is locally compact and the set M is a non-empty compact subset of X unless explicitly stated otherwise.

DEFINITION 5.1 With a given $M \subset X$ we have

1. the set $A_w(M) = \{x \in X: \Lambda^+(x) \cap M \neq \emptyset\}$,
2. the set $A(M) = \{x \in X: \Lambda^+(x) \neq \emptyset \text{ and } \Lambda^+(x) \subset M\}$.

The set $A_w(M)$ is called the region of weak attraction and the set $A(M)$ is called the region of attraction of the set M. Moreover, any point in $A_w(M)$ or in $A(M)$ is said to be weakly attracted or attracted to M respectively.

From this definition it is clear that a point x is weakly attracted to a set M if and only if there is a sequence $\{t_n\}$ in R with $t_n \to +\infty$ and $d(xt_n, M) \to 0$. Also a point x is attracted to a set M if and only if $d(xt, M) \to 0$ as $t \to +\infty$.

The above attractor sets are important in the development of stability theory and the following theorem provides needed information about them. The following lemma is useful.

LEMMA 5.2 *Let* X *be any metric space and let* $x \in X$. *Then for every* $t \in R$,

1. *the set* $\Lambda^+(x) = \Lambda^+(x)t = \Lambda^+(xt)$,
2. *the set* $J^+(x) = J^+(x)t = J^+(xt)$.

PROOF The first equality in statements 1 and 2 follows from the invariance of the sets $\Lambda^+(x)$ and $J^+(x)$ respectively. To see the second equalities, consider the case $\Lambda^+(x)t = \Lambda^+(xt)$. Let $z \in \Lambda^+(x)t$. Then there is a $y \in \Lambda^+(x)$ with $z = yt$ and a sequence $\{t_n\}$, $t_n \to +\infty$, with $xt_n \to y$. Then by the continuity axiom, $xt_n(t) \to yt$. But $xt_n(t) = xt(t_n)$ and since $t_n \to +\infty$, we must have that $yt \in \Lambda^+(xt)$. Thus $\Lambda^+(x)t \subset \Lambda^+(xt)$. The argument is clearly reversible so we also have that $\Lambda^+(xt) \subset \Lambda^+(x)t$. Hence statement 1 is proved. The proof of the second inequality in statement 2 is entirely analogous. //

THEOREM 5.3 *For any given* M, $A_w(M) \supset A(M)$ *and the sets* $A_w(M)$ *and* $A(M)$ *are invariant*.

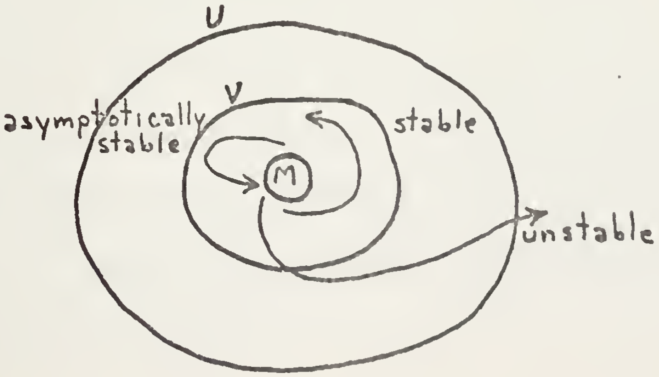
PROOF Let $x \in A(M)$. Then $\Lambda^+(x) \neq \emptyset$ and $\Lambda^+(x) \subset M$. Then clearly $\Lambda^+(x) \cap M \neq \emptyset$. Hence $x \in A_w(M)$. To see the invariance of the sets $A_w(M)$ and $A(M)$, consider the case for $A_w(M)$. Let $x \in A_w(M)$ and let $t \in R$. Now $\Lambda^+(x) \cap M \neq \emptyset$ and from Lemma 5.2, $\Lambda^+(x) = \Lambda^+(xt)$. Hence $\Lambda^+(xt) \cap M \neq \emptyset$ which implies that $xt \in A_w(M)$. Since $t \in R$ was arbitrary, we have that $A_w(M)$ is invariant. The proof of the invariance of $A(M)$ is entirely analogous. //

Using these sets, we are now ready to define the various types of stability to be investigated in this section.

DEFINITION 5.4 A given set M is said to be

1. a weak attractor if $A_W(M)$ is a neighborhood of M,
2. an attractor if A(M) is a neighborhood of M,
3. stable is every neighborhood U of M contains a positively invariant neighborhood V of M,
4. asymptotically stable if it is stable and is an attractor,
5. unstable if it is not stable.

Geometrically, M is stable if given any neighborhood U of M, there is a neighborhood $V \subset U$ such that any trajectory that is within V at time t = 0 remains within V for all $t \geq 0$. If, in addition, all trajectories approach M as t increases, then M is asymptotically stable. Finally M is unstable if every neighborhood of M contains trajectories which are arbitrarily close to M at time t = 0 but which eventually leave the neighborhood for all time. These three cases are illustrated in the diagram below.



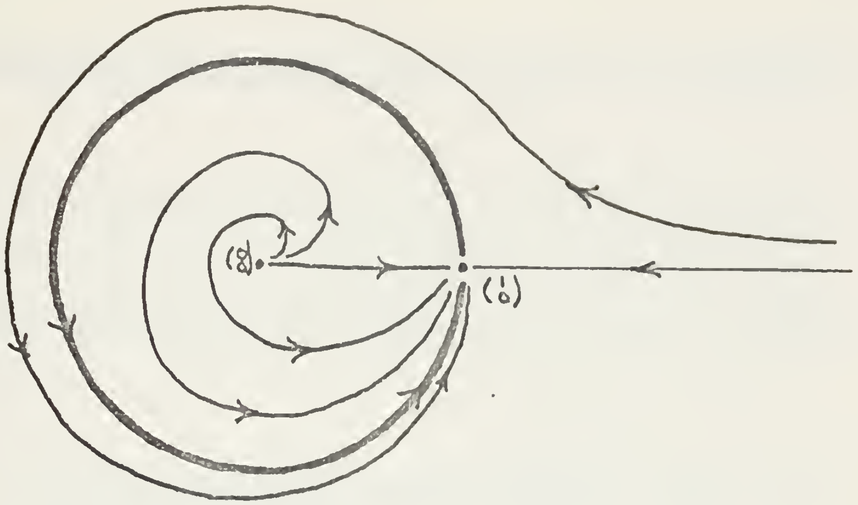
We have seen specific instances of these cases in several of our examples. For instance in the pendulum problem, (Example 2.1), the point $\binom{0}{0}$ is stable; in the damped spring problem (Example 2.2), the point $\binom{0}{0}$ is asymptotically stable and in Example 3.2, if the system is restricted to the interior of the unit circle, the point $\binom{0}{0}$ is unstable. In this latter example, if x is any point on the unit circle, then the point x is a weak attractor although it is not stable. In order to see an example of an attractor which is not stable, consider the following example:

Let the planar dynamical system be defined by the differential equations (in polar coordinates):

$$\dot{r} = r(1 - r),$$

$$\dot{\theta} = \sin^2 (\theta/2).$$

The trajectories of this system are sketched below. They consist of two critical points at $\binom{0}{0}$ and $\binom{1}{0}$, a trajectory on the unit circle with $\binom{0}{0}$ as the positive and negative limit set of all points on the unit circle. All other points have the point $\binom{1}{0}$ as their positive limit set and hence the point $\binom{1}{0}$ is an attractor. It is not stable because every neighborhood will contain part of the unit circle where $\theta > 0$. Hence no positively invariant subset of this neighborhood can exist.



We are now ready to prove some theorems about the concepts just introduced.

THEOREM 5.5 *If* M *is a weak attractor* (*respectively, an attractor*), *then the set* $A_w(M)$ (*respectively*, $A(M)$) *is an open neighborhood of* M.

PROOF Let N denote either of the sets $A_w(M)$ or $A(M)$. Then N is an invariant neighborhood of M. Consequently, bdy N is invariant by Theorem 2.10, is disjoint from M, and is closed. Then for each $x \in N$, $\Lambda^+(x) \cap M \neq \emptyset$, whereas for each $x \in$ bdy N, $\Lambda^+(x) \subset$ bdy N since bdy N is invariant and closed. Since bdy $N \cap M = \emptyset$, we conclude that $N \cap$ bdy $N = \emptyset$. Thus N is open. //

We note that if a set M is stable then it is the intersection of positively invariant neighborhoods and M is therefore positively invariant. Immediately from this we see that *if the singleton* $\{x\}$ *is stable, then* $\{x\}$ *must be*

positively invariant so consequently, x *must be a critical point*.

The next two theorems show different characterizations of stable sets but first we establish the following lemma and its corollary which do not depend on the local compactness of X.

LEMMA 5.6 *Let* X *be an arbitrary metric space*. *Let* $x \in X$ *and* $w \in \Lambda^+(x)$. *Then* $J^+(x) \subset J^+(w)$.

PROOF Given $w \in \Lambda^+(x)$ and any $y \in J^+(x)$, there exist sequences $\{\tau_n'\}$, $\tau_n' \to +\infty$, $x\tau_n' \to w$, and $\{t_n'\}$ and $\{x_n\}$, $x_n \to x$, $t_n' \to +\infty$, $x_n t_n' \to y$. We may assume, if necessary by choosing subsequences, that $t_n' - \tau_n' > n$ for each n. Consider for each fixed k, $k = 1, 2, \ldots$, the sequence $\{x_n \tau_k'\}$. By the continuity axiom $x_n\tau_k' \to x\tau_k'$, $k = 1, 2, \ldots$. We may, therefore, without loss of generality, assume that for each fixed k, $d(x\tau_k', x_n\tau_k') \leq 1/k$ for $n \geq k$. This shows that $x_n\tau_n' \to w$, because $d(w, x_n\tau_n') \leq d(w, x\tau_n') + d(x\tau_n', x_n\tau_n') \leq d(w, x\tau_n') + 1/n$. Now notice that $x_n t_n' = x_n\tau_n'(t_n' - \tau_n')$, and $x_n t_n' \to y$, $x_n\tau_n' \to w$, and $t_n' - \tau_n' > n$. Hence $y \in J^+(w)$. As $y \in J^+(x)$ was arbitrary, we have $J^+(x) \subset J^+(w)$, and the lemma is proved. //

COROLLARY *Given* M *and* $x \in A_w(M)$, *then* $J^+(x) \subset J^+(M) \subset D^+(M)$.

PROOF Indeed for any $x \in A_w(M)$, $\Lambda^+(x) \cap M \neq \emptyset$. Let w be any element in $\Lambda^+(x) \cap M$. Then by the previous lemma, $J^+(x) \subset J^+(w)$. But $J^+(w) \subset J^+(M)$ and we know that $J^+(M) \subset D^+(M)$. Hence $J^+(x) \subset J^+(M) \subset D^+(M)$. //

We now give an important characterization of stability of a set M.

THEOREM 5.7 A set M is stable if and only if $D^+(M) = M$.

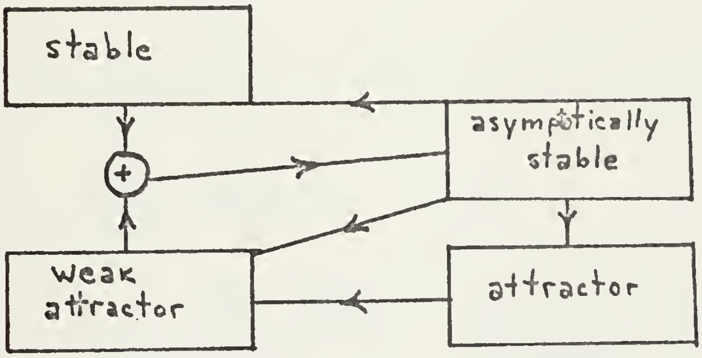
PROOF Let $D^+(M) = M$ and suppose if possible that M is not stable. Then there is an $\varepsilon > 0$, a sequence $\{x_n\}$, and a sequence $\{t_n\}$, with $t_n \geq 0$, $d(x_n,M) \to 0$, and $d(x_n t_n,M) \geq \varepsilon$. We may assume without loss of generality that $\varepsilon > 0$ has been chosen so small that $S[M,\varepsilon]$ and hence $H(M,\varepsilon)$ is compact where $H(M,\varepsilon) = \{x \in X: d(x,M) = \varepsilon\}$. This is possible since X is locally compact. Further, we may assume that $x_n \to x \in M$. We can now choose a sequence $\{\tau_n\}$, $0 \leq \tau_n \leq t_n$, such that $x_n\tau_n \in H(M,\varepsilon)$, $n = 1, 2, \ldots$. Since $H(M,\varepsilon)$ is compact, we may assume that $x_n\tau_n \to y \in H(M,\varepsilon)$. Then clearly $y \in D^+(x) \subset D^+(M)$, but $y \notin M$. This contradiction shows that M is stable. Conversely, assume that M is stable. Then given any neighborhood U of M there is a positively invariant neighborhood V of M with $V \subset U$. Since for any $x \in M$, $D^+(x) \subset \text{cl } WR^+$ for any neighborhood W of x, we get $D^+(x) \subset \text{cl } V$ since V is positively invariant. Thus $D^+(M) \subset \text{cl } U$ for any neighborhood U of M. Hence $D^+(M) \subset \bigcap\{\text{cl } U: U \text{ is a neighborhood of } M\} = M$ as M is compact. Since $M \subset D^+(M)$ always holds, we have $D^+(M) = M$. //

The following remark shows what results when we combine several of the concepts under discussion.

REMARK *If M is stable and is a weak attractor, then M is an attractor and consequently asymptotically stable.*

PROOF Since M is a weak attractor, $A_w(M)$ is a neighborhood of M. We need to show that $A_w(M) \subset A(M)$. Let $x \in A_w(M)$. Then $\Lambda^+(x) \neq \emptyset$ since $\Lambda^+(x) \cap M \neq \emptyset$ and $\Lambda^+(x) \subset J^+(x) \subset D^+(M)$ by the previous corollary. Since M is stable we have $D^+(M) = M$ by Theorem 5.7 and consequently $\Lambda^+(x) \subset M$. Thus $x \in A(M)$ and A(M) is a neighborhood of M. Thus M is an attractor. Since M is stable and is an attractor, M is asymptotically stable.

The various implications that have been proved thus far are summarized in the following diagram.



We would now like to confine our attention to the study of stability for systems of differential equations. We shall begin by considering autonomous systems of constant coefficient linear equations; that is, systems of the form

$$\dot{X} = AX$$

where A is an n×n real valued matrix. The origin in R^n is a critical point for such a system, and when zero is not an eigenvalue for A (so that is when A is non-singular) the origin is the only critical point. The solutions of this system can be described in terms of the eigenvalues of A in the following manner.

Let $\lambda = a + bi$ be a complex eigenvalue for the n×n real matrix A, and let E_λ be an eigenvector in C^n belonging to λ. Then the functions

$$X_1(t) = e^{at}(G_\lambda \cos bt + H_\lambda \sin bt),$$

$$X_2(t) = e^{at}(H_\lambda \cos bt - G_\lambda \sin bt),$$

are linearly independent solutions of $\dot{X} = AX$, where

$$G_\lambda = \frac{E_\lambda + \overline{E_\lambda}}{2} \qquad \text{and}$$

$$H_\lambda = \frac{i(E_\lambda - \overline{E_\lambda})}{2}.$$ [Kreider, Kuller, Ostberg, p. 255]

Now when $a \pm bi$, $b > 0$, are complex conjugate eigenvalues of multiplicity m, this system has 2m linearly independent solutions constructed from functions of the form

$$G_\lambda e^{at} t^k \cos bt$$

and

$$H_\lambda e^{at}t^k \sin bt,$$

where k is an integer such that $0 \leq k \leq m - 1$. Then assuming $t > 0$,

$$||G_\lambda e^{at}t^k \cos bt|| \leq ||G_\lambda||\ e^{at}t^k,$$

and since $||G_\lambda||$ is a positive constant,

$$\lim_{t\to\infty} ||G_\lambda e^{at}t^k \cos bt|| = 0 \quad \text{if } a < 0.$$

On the other hand $||G_\lambda e^{at}t^k \cos bt||$ is unbounded if $a > 0$. Similarly,

$$\lim_{t\to\infty} ||H_\lambda e^{at}t^k \sin bt|| = 0 \quad \text{if } a < 0,$$

but $||H_\lambda e^{at}t^k \sin bt||$ is unbounded if $a > 0$.

Thus every trajectory of this system arising from a pair of complex conjugate eigenvalues for A will approach the origin of R^n as $t \to +\infty$ if the real part of these eigenvalues is negative; it will depart arbitrarily far from the origin, if the real part of these eigenvalues is positive. In the first case, the trajectories exhibit the properties required for asymptotic stability whereas in the second case they exhibit instability. This reasoning clearly applies equally well to trajectories arising from real

eigenvalues for A. It remains to consider the case where A admits a pair of pure imaginary eigenvalues ± bi. If these eigenvalues have multiplicity one, the corresponding solutions of the system are constructed from the functions

$$G_\lambda \cos bt \quad \text{and} \quad H_\lambda \sin bt$$

and the fact that cos bt and sin bt are bounded as $t \to +\infty$ yet do not tend to zero implies that the resulting trajectories have the properties required for stability, but not asymptotic stability. On the other hand, if the multiplicity of ±bi is greater than one, the system has solutions involving functions of the form

$$G_\lambda t^k \cos bt \quad \text{and} \quad H_\lambda t^k \sin bt$$

with $k \geq 1$. When this happens, the origin is unstable since

$$|t^k \cos bt| \quad \text{and} \quad |t^k \sin bt|$$

are unbounded. Thus we have proved the following theorem.

THEOREM 5.8 *If* $\dot{X} = AX$ *is an* n×n *linear autonomous system whose coefficient matrix is nonsingular, then the origin in* R^n *is*

1. *asymptotically stable if the real parts of all of the eigenvalues of* A *are negative;*

2. *stable, but not asymptotically stable, if* A *has at least one pair of pure imaginary eigenvalues of multiplicity one, no pure imaginary eigenvalues of multiplicity exceeding one, and no eigenvalues with positive real parts;*
3. *unstable otherwise.*

Consider the following plane autonomous systems:

$$\begin{array}{lll} 1.\quad \dot{x} = y & 2.\quad \dot{x} = y & 3.\quad \dot{x} = -x \\ \dot{y} = x & \dot{y} = -x & \dot{y} = -y \end{array}$$

The coefficient matrix for the first system is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with eigenvalues $\lambda = \pm 1$. Hence the origin is unstable since one of the eigenvalues has a positive real part.

For the second system,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

This system is precisely that of Example 2.3 and has eigenvalues $\lambda = \pm i$. Thus the origin is stable but not asymptotically stable. This is exactly as we would expect

by recalling the geometry of Example 2.3.

For the third system,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This system has $\lambda = -1$ as its only eigenvalue and thus the origin is asymptotically stable.

The above results can be used to obtain a complete description of the trajectories of a linear plane autonomous system of the form

$$\dot{x} = ax + by$$

$$\dot{y} = cx + dy$$

where the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nonsingular. This description depends upon the nature of the eigenvalues λ_1, λ_2 of A which are the roots of the characteristic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

Since the discriminant of this equation is given by

$$\Delta = (a + d)^2 - 4(ad - bc) = (a - d)^2 + 4bc,$$

it follows that λ_1 and λ_2 will be

1. real and distinct if $\Delta > 0$,
2. real and equal if $\Delta = 0$,
3. complex conjugates if $\Delta < 0$.

Moreover, when λ_1 and λ_2 are the roots of this system, we have $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$. Thus we have

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0,$$

so

$$\lambda_1 + \lambda_2 = a + d, \quad \lambda_1\lambda_2 = ad - bc = \det A$$

Thus when $\Delta \geq 0$, λ_1 and λ_2 will have the same sign if and only if $ad - bc > 0$, and will then be positive or negative according as $a + d$ is positive or negative. On the other hand, if $\Delta < 0$, $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\beta > 0$, and $a + d = 2\alpha$.

In this case λ_1 and λ_2 will be pure imaginary if and only if $a + d = 0$. Otherwise they will have a nonzero real part which agrees in sign with $a + d$.

Now, combining these observations with Theorem 5.8, we have the following description of the origin for a plane

autonomous linear system with constant coefficients.

$$1.\quad \Delta < 0: \begin{cases} a + d < 0 & \text{asymptotically stable} \\ a + d > 0 & \text{unstable} \\ a + d = 0 & \text{stable} \end{cases}$$

$$2.\quad \Delta > 0: \begin{cases} ad - bc > 0 \quad \text{and} \quad a + d < 0 & \text{asymptotically stable} \\ ad - bc > 0 \quad \text{and} \quad a + d > 0 & \text{unstable} \\ ad - bc < 0 & \text{unstable} \end{cases}$$

$$3.\quad \Delta = 0: \begin{cases} ad - bc > 0 \quad \text{and} \quad a + d < 0 & \text{asymptotically stable} \\ ad - bc < 0 & \text{unstable} \end{cases}$$

Thus far we have not discussed the cases where A is singular. These cases are rather uninteresting but for completeness, they are presented here.

Suppose A is the 2×2 zero matrix. Then $\dot{X} = 0$ for all $X \varepsilon R^2$ which implies that each point in R^2 is a critical point and consequently each point is stable since every point within some neighborhood of a point, remains within that neighborhood as $t \to +\infty$.

Suppose A is a 2×2 matrix with zero as its only eigenvalue. Then A must be of the form

$$A = \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}$$

where a is some real number.

To find the eigenvectors X_0 associated with A, we set $AX_0 = 0$. Then all eigenvectors are critical points and are given by the equation

$$0 = \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1 - ax_2$$

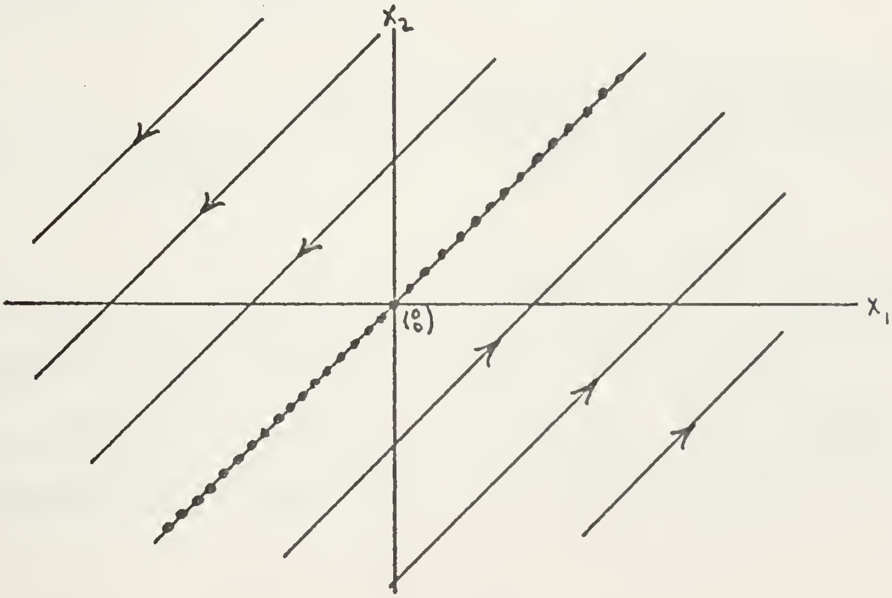
which implies that $x_1 = x_2$. Thus all points on the line through the origin with slope = 1, are critical points.

Now

$$\dot{x}_1 = ax_1 - ax_2$$

$$\dot{x}_2 = ax_1 - ax_2$$

so we have $\dot{x}_1 = \dot{x}_2$ which implies that $x_2 = x_1 + c$ where c is any constant. Thus the remaining trajectories also have slope = 1, and intersect the x_2-axis depending on c. Their directions are shown in the sketch below. It is clear that all critical points are unstable.



Thus far we have studied only linear systems. To study non-linear systems we need the following idea of a linear approximation. Consider the system

$$\dot{X} = F(X)$$

where X is a vector in R^n. We shall show that whenever F is a function of class C^1 (that is, F has continuous first partial derivatives) in a region Ω of R^n containing the origin, then we may replace F by a function of the form

$$JX + G(X),$$

where J is an n×n matrix and G(X) is "small" in comparison with JX when $||X||$ is small. Precisely, we state

DEFINITION 5.9 Let F be as above and let X_0 be a point in Ω. Then a function L = L(X) is said to be a linear approximation to F at X_0 if L is linear on R^n and

$$\lim_{||X-X_0||\to 0} \frac{[F(X) - F(X_0)] - [L(X) - L(X_0)]}{||X - X_0||} = 0.$$

Moreover, the existence and uniqueness of the linear approximation can be guaranteed if F is of class C^1. [Ostberg, Kreider, Kuller pp. 417, 418]. This linear approximation is in fact the n×n Jacobian matrix of F at

every point X_0. Specifically,

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}_{X = X_0}$$

where

$$F(X) = \begin{bmatrix} F_1(X) \\ \cdot \\ \cdot \\ \cdot \\ F_n(X) \end{bmatrix}$$

and $F_i : R^n \to R$ for all i.

To gain a better understanding of the linear approximation, we give some examples.

We find the linear approximation about $\binom{0}{0}$ for each of the following systems:

1. $\dot{x} = P_1(x,y)$

$\dot{y} = P_2(x,y)$

where P_1 and P_2 are polynomials with $P_1(0,0) = P_2(0,0) = 0$. Then $F_1(x,y) = P_1(x,y)$ and $F_2(x,y) = P_2(x,y)$. Now if

$$P_1(x,y) = a_1x + b_1y + c_1xy + a_2x^2 + b_2y^2 + \text{higher-order terms},$$

$$P_2(x,y) = d_1x + e_1y + f_1xy + d_2x^2 + e_2y^2 + \text{higher-order terms},$$

we have

$$\frac{\partial F_1}{\partial x} = a_1 + c_1y + 2a_2x + \text{higher-order terms} \quad \text{and}$$

$$\frac{\partial F_1}{\partial y} = b_1 + c_1x + 2b_2y + \text{higher-order terms}$$

Now evaluating these partials at the point $\binom{0}{0}$, we have

$$\frac{\partial F_1}{\partial x} = a_1 \quad \text{and} \quad \frac{\partial F_1}{\partial y} = b_1$$

Similar results hold for the partials of F_2. Thus we replace $F(X)$ by $JX + G(X)$ where

$$J = \begin{bmatrix} a_1 & b_1 \\ d_1 & e_1 \end{bmatrix}, \text{ and } G(X) = \begin{bmatrix} c_1xy + a_2x^2 + b_2y^2 + \dots \\ f_1xy + d_2x^2 + e_2y^2 + \dots \end{bmatrix}.$$

Hence the linear approximation is given by the following system:

$$\dot{X} = JX.$$

2. $\dot{x} = \sin x + e^y - 1$

$\dot{y} = xy$

Now, sin x and e^y have the following Taylor series expansions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \ldots$$

$$e^y = 1 + y + \frac{y^2}{2!} + \ldots.$$

Hence we have $\dot{x} = x + y - \frac{x^3}{3!} + \frac{y^2}{2!}$ + higher-order terms

$\dot{y} = xy$

and applying the procedure used in the previous example, we have that the approximating system is given by

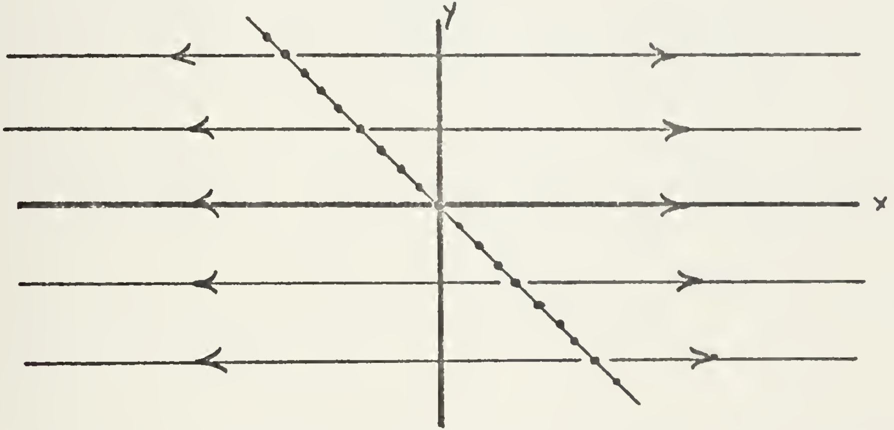
$$\dot{X} = JX$$

where

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

To analyze this system as to stability, we note that the matrix J is singular and hence our theorem does not apply. Thus we must analyze J using the same method we used in

the two previous cases of singular matrices. We see that J has eigenvalues $\lambda = 0,1$ and hence does not fit into the category of the other two examples. However, it is an easy task to determine that all eigenvectors lie on the line of slope -1 through the origin and are all critical points. From the approximating system, it is easily seen that $\dot{y} = 0$ and hence all trajectories not on the line of critical points are parallel to the x-axis. Their directions are easily determined from the equations. They are shown in the following diagram.



It is clear that each critical point is unstable.

One might question now whether the complete theory we developed for linear systems also applies to the approximating systems for non-linear systems. Such is the case for asymptotically stable and unstable systems. That is, if the linear approximating system is unstable, then the non-linear system is unstable. However, no definite conclusion can be made about stable systems. It is recalled that for a system to be stable, it could have no eigenvalues with positive real parts. A non-linear system may have

eigenvalues with small positive real parts which do not appear in its approximating system. These facts are proved in the next section and are presented here just for information.

VI. LIAPUNOV STABILITY

The basic feature of stability theory à la Liapunov is that one seeks to characterize stability and asymptotic stability of a given set in terms of a non-negative real-valued function defined on a neighborhood of the given set. This technique was devised by the Russian mathematician A. A. Liapunov, and is known as Liapunov's direct or second method. It is based on the well-known fact that a physical system loses potential energy in a neighborhood of a point of stable equilibrium. More precisely, a point of stable equilibrium for a physical system is a point at which the potential energy of the system has a local minimum. This fact is known in physics as Lagrange's theorem. We proceed with the following lemma.

LEMMA 6.1 *Let the phase space* X *be arbitrary and let* $K \subset X$. *Let* ϕ *be any continuous real-valued function defined on* K *such that* $\phi(xt) \leq \phi(x)$ *whenever* $x[0,t] \subset K$, $t \geq 0$. *Then if for some* x, $\text{cl}\,(\gamma^+(x)) \subset K$, *we have* $\phi(y) = \phi(z)$ *for every* $y,z \in \Lambda^+(x)$.

PROOF Now there are sequences $\{t_n\}$ and $\{\tau_n\}$ in R such that $t_n \to +\infty$, $\tau_n \to +\infty$, and $xt_n \to y$, $x\tau_n \to z$. We may assume by taking a subsequence that $\tau_n > t_n$ for each n. Then clearly $\phi(xt_n) \geq \phi(x\tau_n)$ since $x\tau_n = xt_n(\tau_n - t_n)$, $\tau_n - t_n > 0$, and $x_n t_n[0,\tau_n - t_n] \subset K$. Thus proceeding to

the limit we have by the continuity of ϕ, $\phi(y) \geq \phi(z)$. A similar argument shows that $\phi(z) \geq \phi(y)$. Thus the lemma is proved. //

The best known result on asymptotic stability is the following.

THEOREM 6.2 *A compact set* $M \subset X$ *is asymptotically stable if and only if there exists a continuous real-valued function* E *defined on a neighborhood* N *of* M *such that*

1. *if* $x \in M$, *we have* $E(x) = 0$ *and if* $x \notin M$, *we have* $E(x) > 0$,
2. *for* $x \notin M$, $t > 0$ *and* $x[0,t] \subset N$, *we have* $E(xt) < E(x)$.

PROOF Assume that a function E as required is given. Choose $a > 0$ such that $S[M,a] \subset N$ and is compact. Let $m = \min\{E(x) : x \in H(M,a)\}$ where $H(M,a) = \{x \in X : d(M,x) = a\}$. Such a minimum exists because E is continuous and $H(M,a)$ is a closed and bounded set. By statement 1 and the continuity of E we have $m > 0$. Set $K = \{x \in S[M,a] : E(x) \leq m\}$. Then K is compact and because of statement 2, K is positively invariant. This establishes that M is stable as K is a positively invariant neighborhood of M. To see that M is an attractor, choose any compact positively invariant neighborhood K of M with $K \subset N$. Then for any $x \in K$, $\gamma^+(x) \subset K$ and since K is compact, $\mathrm{cl}\,(\gamma^+(x)) \subset K$. Hence $\mathrm{cl}\,(\gamma^+(x))$ is compact and we have

$\emptyset \neq \Lambda^+(x) \subset K$, and Lemma 6.1 shows that E is constant on $\Lambda^+(x)$. But this shows by statement 2 that $\Lambda^+(x) \subset M$. Thus M is an attractor and, consequently, asymptotically stable.

Conversely, let M be asymptotically stable and A(M) its region of attraction. For each $x \in A(M)$ define

$$\phi(x) = \sup\{d(xt,M): t \geq 0\}.$$

Indeed $\phi(x)$ is defined for each $x \in A(M)$ because if $d(x,M) = a$, then there is a $T > 0$ with $x[T,+\infty) \subset S(M,a)$. This is because $x \in A(M)$. Thus

$$\phi(x) \equiv \sup\{d(xt,M): 0 \leq t \leq T\}.$$

Since $d(xt,M)$ is a continuous function of t, $\phi(x)$ is defined. Now $\phi(x)$ has the following properties: $\phi(x) = 0$ for $x \in M$, $\phi(x) > 0$ for $x \notin M$, and $\phi(xt) \leq \phi(x)$ for $t \geq 0$. This is clear when we remember that M is stable hence positively invariant, and that A(M) is invariant. Thus if $\phi(x)$ is defined for any $x \in A(M)$ it is defined for all xt with $t \in R^+$. We further claim that this $\phi(x)$ is continuous in A(M). Stability of M implies continuity of $\phi(x)$ on M. For $x \notin M$, this proof requires a knowledge of uniform attractors and it will not be presented here. However, this function is continuous in A(M) [Bhatia, Szego p. 67]. This function may not be strictly decreasing along

trajectories in A(M) which are not in M and so may not satisfy statement 2 of the theorem. However we can obtain the desired function by setting

$$E(x) = \int_0^\infty \phi(x\tau)\exp(-\tau)\, d\tau.$$

Clearly E(x) is continuous and satisfies statement 1 in A(M). To see that E(x) satisfies statement 2, let $x \notin M$ and $t > 0$. Then $E(xt) \leq E(x)$ holds because $\phi(xt) \leq \phi(x)$ holds. To rule out $E(xt) = E(x)$, observe that in this case we must have $\phi(x(t + \tau)) \equiv \phi(x\tau)$ for all $\tau \geq 0$. Thus in particular, letting $\tau = 0, t, 2t, \ldots$ we get $\phi(x) = \phi(x(nt))$, $n = 1, 2, \ldots$ But asymptotic stability of M implies that for $x \in A(M)$, $d(xt,M) \to 0$ as $t \to +\infty$. Thus $\phi(x(nt)) \to 0$ as $n \to \infty$, as $\phi(x)$ is continuous. This shows that $\phi(x) = 0$. But as $x \notin M$, we must have $\phi(x) > 0$ and we have a contradiction. Hence $E(xt) < E(x)$ for $x \notin M$ and $t > 0$. Thus statements 1 and 2 are satisfied. //

This theorem says nothing about the size of the region of attraction of M. Thus if a function E(x) is known to exist in a neighborhood N of M, we need not have either $N \subset A(M)$ or $A(M) \subset N$. This means that the above theorem cannot immediately be stated as a theorem on global asymptotic stability. The following example helps to illustrate this point.

EXAMPLE 6.3 Consider a dynamical system defined in the x,y-plane by the differential equations

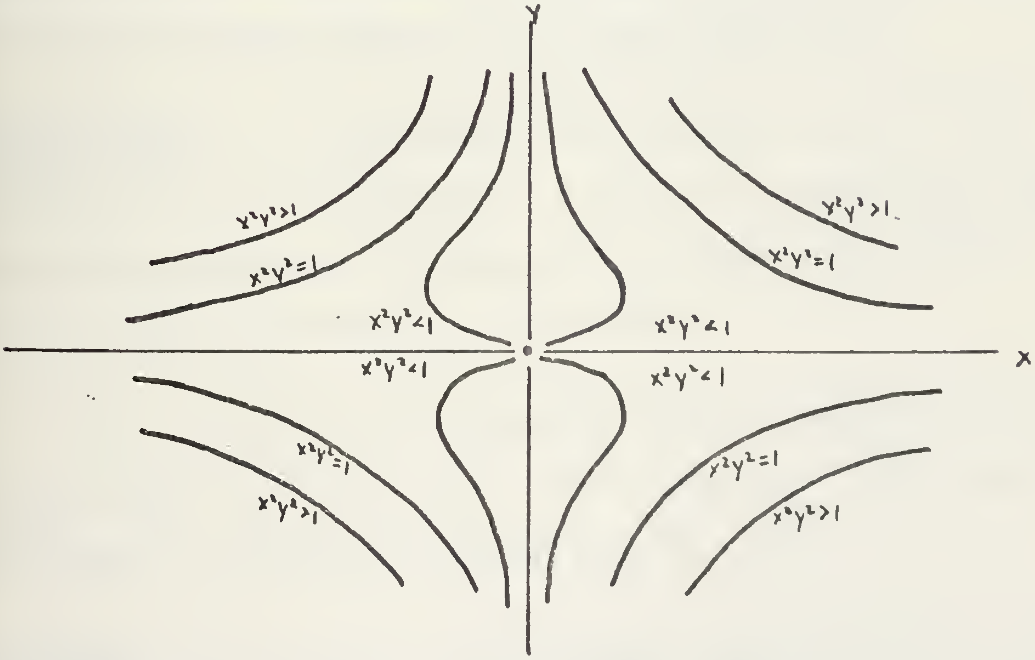
$$\dot{x} = f(x,y),$$

$$\dot{y} = -y,$$

where

$$f(x,y) = \begin{cases} x \text{ is } x^2y^2 \geq 1 \\ 2x^3y^2 - x \text{ if } x^2y^2 < 1. \end{cases}$$

The trajectories of this system are sketched below.



The origin $\binom{0}{0}$ is asymptotically stable, with the set $\{\binom{x}{y}: x^2y^2 < 1\}$ as its region of attraction. Consider now the function

$$E(x,y) = y^2 + \frac{x^2}{1 + x^2} \quad .$$

This function satisfies the conditions of Theorem 6.2 in the whole plane. To see this we note $\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}$ which implies $\dot{E}(x,y) = \frac{\partial E}{\partial x}f(x,y) - y\,\frac{\partial E}{\partial y}$. It can be verified that $\dot{E}(x,y) < 0$ for all $\binom{x}{y} \neq \binom{0}{0}$, which implies that E satisfies the conditions of the theorem in every neighborhood of the origin. But not every neighborhood of the origin is contained in the region of attraction, nor does it contain the region of attraction.

In view of this example, we wish to establish conditions necessary to ensure global asymptotic stability. This is done by starting with the following definition.

DEFINITION 6.4 A continuous real-valued function E(x) defined on a set $N \subset X$ will be said to be *uniformly unbounded* on N if given any $a > 0$ there is a compact set $K \subset N$, $K \neq N$, such that $E(x) \geq a$ for $x \not\subset K$.

The following three theorems will be stated without proof since the proofs are very laborious. The proofs may be found in Bhatia and Szego, pp. 70 - 73.

THEOREM 6.5 *Let* $M \subset X$ *be a compact asymptotically stable set and let* M *be invariant. Then there exists a continuous uniformly unbounded function* E(x) *defined on* A(M) *such that*

1. *for* $x \in M$, $E(x) = 0$ *and for* $x \notin M$, $E(x) > 0$,
2. *for all* $x \in A(M)$ *and* $t \in R$, $E(xt) = e^{-t}E(x)$.

THEOREM 6.6 *If* $M \subset X$ *is any compact asymptotically stable set, there exists a continuous, uniformly unbounded function* $E(x)$ *on* $A(M)$ *such that*

1. *for* $x \in M$, $E(x) = 0$ *and for* $x \notin M$, $E(x) > 0$,
2. *for* $x \notin M$ *and* $t > 0$, $E(xt) < E(x)$.

THEOREM 6.7 *Let* $M \subset X$ *be compact and let there exist a continuous uniformly unbounded function* $E(x)$ *defined on an open neighborhood* N *of* M *such that*

1. *for* $x \in M$, $E(x) = 0$ *and for* $x \notin M$, $E(x) > 0$,
2. *for* $x \notin M$, $t > 0$ *and* $x[0,t] \subset N$, $E(xt) < E(x)$.

Then M *is asymptotically stable and* $N \subset A(M)$. *If in addition, any condition guaranteeing the invariance of* N *holds, then* $N = A(M)$.

These three theorems are very similar in their conclusions but we can see that the hypotheses are becoming more general. The next results on global asymptotic stability follow from these theorems.

THEOREM 6.8 *A compact invariant set* $M \subset X$ *is globally asymptotically stable if and only if there exists a continuous uniformly unbounded function* $E(x)$ *defined on* X *such that*

1. *for* $x \in M$, $E(x) = 0$ *and for* $x \notin M$, $E(x) > 0$,
2. *for all* $x \in X$ *and* $t \in R$, $E(xt) = e^{-t}E(x)$.

PROOF The sufficiency follows from Theorem 6.7 since X is an invariant neighborhood of M. The necessity follows from Theorem 6.5. //

THEOREM 6.9 *A compact set* $M \subset X$ *is globally asymptotically stable if and only if there exists a continuous uniformly unbounded function* $E(x)$ *defined on* X *such that*

1. *for* $x \in M$, $E(x) = 0$ *and for* $x \notin M$, $E(x) > 0$,
2. *for* $x \notin M$ *and* $t > 0$, $E(xt) < E(x)$.

PROOF The sufficiency follows from Theorem 6.7 and the necessity follows from Theorem 6.6. //

These theorems allow us to determine whether a set is asymptotically stable or not without having to solve the system. However, if a set is asymptotically stable, it may not be a trivial problem to display the continuous uniformly unbounded Liapunov function. As in the last section, we now focus our attention directly on systems of differential equations in order to apply some of the above theory.

Thus we return to the general case of an autonomous system

$$\dot{X} = F(X)$$

defined in a region Ω of R^n with a critical point at X_0 in Ω. Without loss of generality, we may assume that the

critical point is the origin $\tilde{0}$ of R^n for if it is not, a simple change of variable $X^* = X - X_0$ will translate the origin to the point X_0 without changing the form of the system.

Again we seek a real-valued function $E(X)$ of class C^1 defined on some region Ω of the origin such that $E(X) \geq 0$ and $E(X) = 0$ if and only if $X = \tilde{0}$. We will refer to E as the energy function because of our previous physical interpretation. If $X = X(X_0,t)$ is the solution of

$$\dot{X} = F(X)$$

$$X(0) = X_0,$$

the time rate of change of energy, $\frac{\partial E}{\partial t}$ along the trajectory defined by this solution is

$$\nabla E \cdot \dot{X} = \nabla E \cdot F = \sum_{i=1}^{n} \frac{\partial E}{\partial x_i} F_i$$

Thus if this expression is negative in Ω, a particle moving along any trajectory defined by the system will be dissipating energy as it intersects the surface $E(X) = c$. Consequently a particle which enters the region enclosed by such a surface can never gather enough energy to escape, and its trajectory must remain in that region. This implies that the origin will be asymptotically stable, for the given system.

These remarks motivate the following definition.

DEFINITION 6.10 Let Ω be a region of R^n containing the origin, and let $E = E(X)$ be a real-valued function of class C^1 in Ω. Then E is said to be positive definite if

1. for all X in Ω, $E(X) \geq 0$ and
2. the equality $E(X) = 0$ holds if and only if $X = \tilde{0}$.

If in addition we have $\nabla E \cdot F = \sum_{i=1}^{n} \frac{\partial E}{\partial x_i} F_i \leq 0$ everywhere in Ω, then E is said to be the Liapunov function for the autonomous system $\dot{X} = F(X)$.

In terms of this definition, we can now form some conclusions about the stability of the origin.

THEOREM 6.11 The origin is a point of stable equilibrium for

$$\dot{X} = F(X),$$

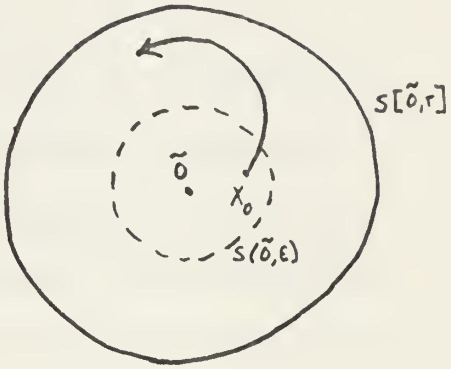
$$F(X_0) = \tilde{0}$$

provided there exists a Liapunov function E for the system.

PROOF Assume the existence of the Liapunov function for the system and consider the set $S[\tilde{0},r] = \{X \in R^n: d(\tilde{0},X) \leq r\}$. Let r be so chosen that $S[\tilde{0},r]$ lies in Ω. Since $S[\tilde{0},r]$ is compact and E is continuous and positive definite in Ω, it follows that E

achieves its minimum value on $S[\tilde{0},r]$. Let m be this minimum value. Clearly m is positive.

Again since E is continuous, there exists an $\varepsilon > 0$ such that $||E(X) - E(\tilde{0})|| < m$ whenever $||X - \tilde{0}|| < \varepsilon$, or simply stated, $E(X) < m$ whenever $||X|| < \varepsilon$. Consider the set $S(\tilde{0},\varepsilon)$. Let X_0 be any point in $S(\tilde{0},\varepsilon)$ and let $X(X_0,t)$ be the unique solution of $\dot{X} = F(X)$ whose trajectory passes through X_0 at time $t = 0$.



Now we claim that the trajectory must remain within $S[\tilde{0},r]$ for all $t > 0$. Assume that this is not the case. Then $||X(X_0,t_1)|| = r$ for some $t_1 > 0$, since the trajectory is a connected set. Thus $E(X(X_0,t_1)) \geq m$ but this implies that E is increasing with time and that $\frac{\partial E}{\partial t} > 0$. Now $\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x_i} \frac{dx_i}{dt} = \nabla E \cdot F$. Hence $\nabla E \cdot F > 0$. But this contradicts the assumption that $\nabla E \cdot F \leq 0$ in Ω, since this inequality implies that E is a non-increasing function of t along the trajectory in question. Thus the trajectory remains within $S[\tilde{0},r]$ for all $t > 0$ and hence the origin is stable since $S[\tilde{0},r]$ is a positively invariant neighborhood of the origin. //

This argument suggests that if E actually decreases along trajectories in a neighborhood of the origin, then

the origin would be asymptotically stable. The next theorem uses this fact.

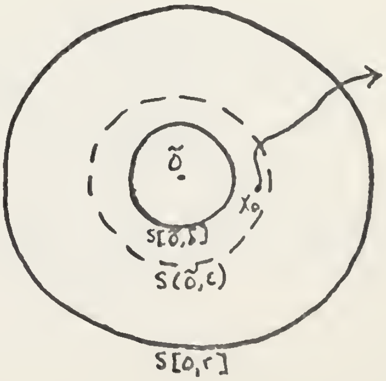
THEOREM 6.12 *If* E *is a Liapunov function for* $\dot{X} = F(X)$ *with the property that* $-\nabla E \cdot F$ *is positive definite in* Ω, *then the origin is asymptotically stable*.

PROOF We have seen that there exists an $\varepsilon > 0$ with the property that E is a non-increasing function along any trajectory $X(X_0,t)$ with $||X_0|| < \varepsilon$. Since E is positive definite in Ω, we claim that E approaches a non-negative limit E_0 along this trajectory as $t \to \infty$. To see this, we form the sequence $\{E(X(X_0,t_n))\}$ where $t_n \to \infty$.

Now $E(X(X_0,t_k))$ assumes values in $[0,E(X_0)]$. Since $[0,E(X_0)]$ is a compact set, every sequence in $[0,E(X_0)]$ has a subsequence which converges to a point E_0 in $[0,E(X_0)]$. Moreover E_0 is non-negative.

We will be done if we show that the limit is zero when $-\nabla E \cdot F$ is positive definite in Ω. Then the fact that E is zero only at the origin will imply that the trajectory approaches the origin with increasing time.

On the contrary, suppose $E_0 > 0$. Then since E is continuous there exists a $\delta > 0$ such that $||E(X) - E(\tilde{0})|| < E_0$ when $||X|| < \delta$. Consider the set $S[\tilde{0},\delta]$. Now the trajectory $X(X_0,t)$ cannot enter the sphere $S[\tilde{0},\delta]$ for if it did, then



$E(X(X_0,t_k)) < E_0$ for some $k > 0$. But since E is a non-increasing function, it could never leave $S[\tilde{0},\delta]$ and thus could not approach E_0 as a limit.

Now since F is continuous and E is of class C^1 in Ω, $-\nabla E\cdot F$ is continuous and hence assumes its minimum value on the "annular" region $\delta \leq ||X|| < r$ where r is as in the proof of the previous theorem. This fact is true because this "annular" region is compact. Let m be this minimum value of $-\nabla E\cdot F$. Since $-\nabla E\cdot F$ is positive definite, it follows that $m > 0$. Thus

$$\frac{\partial E(X(X_0,t))}{\partial t} \leq -m \quad \text{for all } t \geq 0$$

Now

$$E(X(X_0,t)) - E(X(X_0,0)) = \int_0^t \frac{\partial E(X(X_0,\tau))}{\partial \tau}\, d\tau$$

by the fundamental theorem of the calculus. But

$$\int_0^t \frac{\partial E(X(X_0,\tau))}{\partial \tau}\, d\tau \leq \int_0^t -m d\tau = -mt.$$

Thus we have $E(X(X_0,t)) \leq E(X_0) - mt$.

Since the right hand side approaches $-\infty$ as $t \to \infty$, this implies that $E(X(X_0,t)) < 0$ for all $t > t_0$ where $t_0 = \frac{E(X_0)}{m}$ but this contradicts the fact that E is positive

definite in Ω. Thus $E_0 = 0$ and the origin is asymptotically stable. //

Thus far we have presented theorems only on stability or asymptotic stability of the origin. However it is also important to know under what conditions the origin is unstable. Consequently we present the following instability theorem known as Liapunov's First Instability Theorem.

THEOREM 6.13 *Let* E *be a real-valued function of class* C^1 *in* Ω *and suppose that* $E(\tilde{0}) = 0$, $\dot{E}$ *is positive definite and* E *is able to assume positive values arbitrarily near the origin. Then the origin is unstable.*

PROOF Let $S[\tilde{0},r]$ and $S[\tilde{0},\varepsilon]$ be spheres about the origin such that $\varepsilon < r$ and $S[\tilde{0},r] \subset \Omega$. Since $E(X)$ and the first partials of $E(X)$ are continuous in Ω, we have that $E(X)$ is bounded in $S[\tilde{0},r]$. Hence $E(X) \leq k$ for all $X \in S[\tilde{0},r]$. Choose any $X_0 \in S[\tilde{0},\varepsilon]$ such that $E(X_0) > 0$ and let $X(X_0,t)$ be the unique solution of $\dot{X} = F(X)$ whose trajectory passes through X_0 at time $t = 0$.

Now $X(X_0,t)$ must eventually cross bdy $S[\tilde{0},r]$ for if not then $E(X(X_0,t))$ approaches a limit p in $[E(X_0),k]$. Since $E(X(X_0,t)) \to k$ as $t \to \infty$, it follows that $\dot{E}(X(X_0,t)) \to 0$. Now $\dot{E}(X) = \nabla E \cdot F$ and since E and F are continuous, we have that $\dot{E}(X)$ is continuous. Then since $\dot{E}(X(X_0,t)) \to 0$, it follows that $X(X_0,t) \to \tilde{0}$ since E is positive definite. But this is a contradiction since if $X(X_0,t) \to \tilde{0}$ as $t \to \infty$,

it must follow that $E(X(X_0,t)) \to 0$ because E is continuous. Thus since $E(X_0) > 0$ we have $\dot{E}(X(X_0,t)) < 0$ but this is a contradiction. Hence $X(X_0,t)$ must eventually cross the boundary of $S[\tilde{0},r]$ and thus the origin is unstable. //

The important feature of these theorems is that they enable us to determine the stability or instability of critical points without actually solving the system. This is particularly important in cases where the solutions are impossible to obtain in closed form or are difficult to analyze. We can still gain valuable information on their stability provided that we can construct a Liapunov function for the system. Although there is no general method for constructing these functions, we shall see that this is fairly easy to do in many cases. To solidify our understanding of this notion, we consider the following example.

EXAMPLE The function $E(x,y) = x^2 + y^2$ is a Liapunov function for the system

$$\dot{x} = -x + x^2y,$$

$$\dot{y} = -y + xy^2.$$

Clearly, E is positive definite and of class C^1 in the entire x,y-plane. Moreover, since

$\nabla E = \frac{\partial E}{\partial x} e_1 + \frac{\partial E}{\partial y} e_2 = 2xe_1 + 2ye_2$ and

$$F(x,y) = (-x + x^2y)e_1 + (-y + xy^2)e_2$$

we have

$$\nabla E \cdot F = -2(x^2 + y^2) + 2x^3y + 2xy^3 = 2(x^2 + y^2)(xy - 1).$$

Hence $-\nabla E \cdot F$ is positive definite in the region $xy < 1$, and E satisfies all the requirements of Theorem 6.12 in this region. Thus the origin is an asymptotically stable critical point for this system.

We shall now describe a method of constructing a Liapunov function. To do so we begin by constructing a Liapunov function for the constant coefficient linear autonomous system

$$\dot{X} = AX$$

when all the eigenvalues of A have negative real parts. It is noted that we previously determined that the origin is asymptotically stable for this system. Thus we will not gain anything new from the construction but it will allow us to extend some of our earlier results to non-linear systems. The construction goes as follows.

Let $e_1, \ldots, e_n$ denote the standard basis vectors in R^n and, for each i, $1 \leq i \leq n$, let

$$X_i(t) = X_i(e_i,t) = \begin{bmatrix} x_{i1}(t) \\ \cdot \\ \cdot \\ \cdot \\ x_{in}(t) \end{bmatrix}$$

denote the solution of this system which satisfies the initial condition $X_i(0) = e_i$. Then if $Y = y_1e_1 + \ldots + y_ne_n$ is any vector in R^n, the function

$$X(Y,t) = y_1X_1(t) + \ldots + y_nX_n(t)$$

is the solution of the system which satisfies the initial condition $X(0) = Y$. We now set

$$E(Y) = \int_0^\infty ||X(Y,t)||^2 \, dt,$$

and note that if this integral converges, then $E(Y)$ is positive definite. It is clear that $E(Y) \geq 0$. Also if $Y = \tilde{0}$, then for all i, $1 \leq i \leq n$, we have $y_i = 0$ and hence $X(\tilde{0},t) = 0$. Thus $E(\tilde{0}) = 0$. Now suppose $E(\tilde{0}) = 0$. This implies that $X(Y,t) = 0$. Hence $y_i = 0$ for all i, $1 \leq i \leq n$ since otherwise $X_i(t)$ must be zero for some i. But then $X_i(0) \neq e_i$. Thus we must have that $Y = \tilde{0}$.

The proof that $E(Y)$ is defined for all Y in R^n can be found in Ostberg, Kreider, Kuller, p. 415.

Finally we show that E is in fact a Liapunov function for this system. We compute the value of $E(X(X_0,t))$ where

$X(X_0,t)$ is the solution of $\dot{X} = AX$ satisfying $X(X_0,0) = X_0$. Hence we have

$$E(X(X_0,t)) = \int_0^\infty ||X[X_0,t),s]||^2 ds = \int_0^\infty ||X(X_0,s + t)||^2\, ds$$

$$= \int_t^\infty ||X(X_0,u)||^2\, du.$$

Hence

$$\frac{\partial E(X(X_0,t))}{\partial t} = \frac{\partial}{\partial t} \int_t^\infty ||X(X_0,u)||^2\, du = -||X(X_0,t)||^2$$

as required since $\nabla E \cdot F = \partial E/\partial t$ along the trajectories of $\dot{X} = AX$. Moreover, since the real-valued function $||X||^2$ is positive definite on R^n, E satisfies all the hypotheses of Theorem 6.12 and again we have proved that the origin is asymptotically stable.

We now use this method to construct a Liapunov function for the following linear system:

$$\dot{x} = -x$$

$$\dot{y} = -2y.$$

This system has the coefficient matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$

with eigenvalues -1, -2. In this case the solution $X_1(t)$ and $X_2(t)$ such that $X_1(0) = e_1$, $X_2(0) = e_2$ are given by

$$X_1(t) = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$$

Thus if $Y = xe_1 + ye_2$,

$$\begin{aligned} E(Y) &= \int_0^\infty ||xe^{-t}e_1 + ye^{-2t}e_2||^2 \, dt \\ &= \int_0^\infty (x^2e^{-2t} + y^2e^{-4t}) \, dt \\ &= \frac{x^2}{2} + \frac{y^2}{4} . \end{aligned}$$

The fact that this function satisfies the hypotheses of Theorem 6.12 is easily verified and thus the origin is asymptotically stable.

We now wish to extend these results to non-linear systems. For this, we recall the notion of the linear approximation which was developed in the last section. We recall that in the system

$$\dot{X} = F(X),$$

the function F may be replaced by a function of the form $JX + G(X)$, where J is the n×n Jacobian matrix of F and $G(X)$ is small in comparison with JX when $||X||$ is small.

Now let F be a function of class C^1 in a region Ω of R^n containing the origin, and suppose that $F(\tilde{0}) = \tilde{0}$ and the real parts of all of the eigenvalues of the Jacobian matrix J of F evaluated at the origin are negative. We now show that under these conditions the Liapunov function $E = E(X)$ for the constant coefficient linear system $\dot{X} = JX$ that has been previously constructed is also a Liapunov function for the non-linear system

$$\dot{X} = F(X).$$

Now by our earlier results, we know that $\nabla E \cdot (JX) \leq -||X||^2$. Hence

$$\begin{aligned} \nabla E(X) \cdot F(X) &= \nabla E(X) \cdot [JX + G(X)] \\ &= \nabla E(X) \cdot JX + \nabla E(X) \cdot G(X) \\ &\leq -||X||^2 + ||\nabla E(X)||\ ||G(X)||. \end{aligned}$$

This last statement is a consequence of the Cauchy-Schwarz inequality. Now we also know that $||\nabla E(X)|| \leq 2k||X||$, since ∇E is a linear transformation on a finite-dimensional space. Thus we have

$$\nabla E(X) \cdot F(X) \leq -||X||^2 + 2k||X||\ ||G(X)||.$$

We now use the fact from the definition of a linear approximation that $\lim_{||X||\to 0} \frac{F(X) - JX}{||X||} = 0$. But then it follows that $\lim_{||X||\to 0} \frac{G(X)}{||X||} = 0$. Hence we can find a sphere $S[\tilde{0},\delta]$ about the origin in R^n such that

$$||G(X)|| \leq \frac{1}{4k}\,||X||$$

for all $X \varepsilon S[\tilde{0},\delta]$. Then within this sphere we have

$$\nabla E(X)\cdot F(X) \leq -||X||^2 + (2k||X||)(\tfrac{1}{4k}||X||) = -\tfrac{1}{2}||X||^2$$

and indeed E is a Liapunov function for the system $\dot{X} = F(X)$. Moreover, since this inequality also implies that $-\nabla E\cdot F$ is positive definite in $S[\tilde{0},\delta]$, the hypotheses of Theorem 6.12 are satisfied and we have established the following result.

THEOREM 6.14 *Let* $F = F(X)$ *be of class* C^1 *in a region of* R^n *containing the origin, and suppose that* $F(\tilde{0}) = \tilde{0}$. *Then the origin is asymptotically stable for the system* $\dot{X} = F(X)$ *whenever all of the eigenvalues of the Jacobian matrix of* F *evaluated at the origin have negative real parts.*

As before, there is a companion result when all of the eigenvalues of the Jacobian matrix of F have positive real parts. In that case the origin is a point of complete instability for $\dot{X} = F(X)$.

These statements verify the comments in Section V about the conclusions that may be drawn from a linear approximation. Finally we show by means of an example that no definite conclusions can be made from a system whose linear approximation shows that the origin is a stable critical point.

EXAMPLE Consider the plane autonomous system

$$\dot{x} = y - xf(x,y)$$

$$\dot{y} = -x - yf(x,y)$$

where f has a convergent power-series expansion in a neighborhood of the origin in R^2 and $f(0,0) = \tilde{0}$.

Taking the linear approximation, we have

$$\dot{X} = JX$$

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

In this case J has eigenvalues $\pm i$ so by Theorem 5.8, the origin is stable for the system $\dot{X} = JX$.

Now let $E(x,y) = x^2 + y^2$. Then

$$\nabla E \cdot F = \binom{2x}{2y} \cdot \begin{pmatrix} (y - xf(x,y)) \\ (-x - yf(x,y)) \end{pmatrix} = -2(x^2 + y^2)f(x,y).$$

Hence for this system the origin is

1. stable when $f(x,y) \geq 0$ in some neighborhood of $\binom{0}{0}$;
2. asymptotically stable when $f(x,y)$ is positive definite in a neighborhood of $\binom{0}{0}$;
3. unstable when $f(x,y) < 0$ in every neighborhood of $\binom{0}{0}$.

This completes our study of Liapunov functions and stability theory in general. At this point, the reader should have sufficient background in this theory to be able to read and understand further results in this area. Specifically recommended for further reading is Chapter VIII, C^1 -Liapunov Functions for Ordinary Differential Equations, in Bhatia and Szego.

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ABSTRACT

This thesis investigates stable, asymptotically stable and unstable dynamical systems from a topological point of view with direct application and interpretation to systems of differential equations. Knowledge of topology is not a prerequisite. Specifically, such concepts as Poisson and Liapunov stability as well as parallelizable and dispersive systems are investigated.

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